

Numerical Solution of Dirichlet Boundary-Domain Integro-Differential Equation with Less Number of Collocation Points

**Nurul Akmal Mohamed^{a*}, Nurul Farihan Mohamed^b,
Nurul Huda Mohamed^c, and Mohd Rozni Md Yusof^d**

^{a, c} Mathematics Department, Faculty of Science & Mathematics
35900 Universiti Pendidikan Sultan Idris, Proton City
Tanjong Malim, Perak, Malaysia

^{*}Corresponding author

^b Mathematics Department, Faculty of Science, 81310, Universiti Teknologi
Malaysia, Johor Bahru, Johor, Malaysia

^d Physics department, Faculty of Science & Mathematics, 35900 Universiti
Pendidikan Sultan Idris, Proton City, Tanjong Malim, Perak, Malaysia

Copyright © 2016 Nurul Akmal Mohamed et al. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we show that we have two approaches in implementing of Boundary-Domain Integro-Differential Equation (BDIDE) associated to Dirichlet Boundary Value Problem (BVP) for an elliptic Partial Differential Equation (PDE) with a variable coefficient. One way is by choosing the collocation points at all nodes i.e. on the boundary and interior domain. The other approach is choosing the collocation points for the interior nodes only. We present the numerical implementation of the BDIDE associated to Dirichlet BVP for an elliptic PDE with a variable coefficient by using the second approach. The BDIDE is consisting of several integrals that exhibit singularities. Generally, the integrals are evaluated by using Gauss- Legendre quadrature formula. Our numerical results show that the use of semi-analytic method gives high accuracy results. The discretized BDIDE yields a system of equations. We then apply by a direct method i.e. LU decomposi-

tion method to solve the systems of equations. In all the test domains, we present the relative errors of the solutions and the relative error for the gradient.

Keywords: Direct united boundary-domain integro-differential equation, Dirichlet problem, partial differential equation, semi-analytic integration method

1. Introduction

Almost all physical phenomena can be depicted by either ordinary or partial differential equations (ODE or PDE) together with the boundary conditions. Some of the well-known PDEs are wave equation, heat diffusion equation, Helmholtz equation, Maxwell equations and Schrödinger equation.

Some numerical methods in solving PDE are finite difference method, finite element method, boundary element method (BEM) and boundary-domain element method (BDEM).

The BEM and BDEM are formulated as integral equations. The difference between BEM and BDEM is that the BEM is for solving Boundary Integral Equations (BIEs) whereas the BDEM is the numerical method in solving Boundary Domain Integral Equation and Boundary-Domain Integro-Differential Equations (BDIDEs). See e.g. [11].

[1], [13] and [5] provide an excellent discussions on BEM. However, in order to use BEM, the fundamental solution is necessary which is not generally available for PDE with variable coefficient $a(x)$.

A parametrix method is the method to establish fundamental solutions for elliptic PDE with variable coefficients, see in [6]. By employing a parametrix (Levi function) as an alternative to a fundamental solution, it enables us to reduce a BVP for PDE with variable coefficient not to BIEs but to Boundary-Domain Integral Equations (BDIEs) (see, e.g. [4], [10] and [14]).

The classical works in [6], [4], [10] and [14] were carried on the indirect BDIEs for Dirichlet and Neumann BVPs. Whereas, in this paper, we will deal with the direct BDIEs for Dirichlet BVP.

In this paper, we will deal with the direct BDIDE for Dirichlet problem. For the discussions on the direct BDIDE, see e.g. [7], [3] and [9].

There are two types of direct BDIDEs. The first type is a segregated BDIDE that is when the unknown boundary functions and unknown functions inside the domain are regarded as formally unrelated to each other. The second type is the united BDIDE that is when the unknown boundary functions and the unknown functions inside the domain are related to each other. See e.g. [7] for the analysis of direct united BDIDEs. See e.g. [2] and [3] for the discussions on segregated BDIDE.

Mikhailov & Mohamed in [8] and [9] implemented the numerical computations for BDIE associated to Neumann BVP for PDE with variable coefficient. The spectral properties for BDIE's operator associated with Neumann problem is given

in [9] by providing numerically the obtained several highest maximal eigen-values of the Neumann BDIE's operator.

In [12], the semi-analytic integration method for direct united BDIDE related to Dirichlet problem was constructed. This semi-analytic integration method is a method to reduce the integration error for the boundary integral that exhibits a logarithmic singularity as in the parametrix $P(x, y)$. However, no numerical experiments are reported in [12].

In this paper, we deal with the numerical implementation of the Dirichlet BDIDE of BVP for PDE with variable coefficients. The semi-analytic method as proposed in [12] is also used in this paper.

2. The Dirichlet BDIDE

We consider the linear second-order elliptic PDE with a variable coefficient $a(x)$ as follows.

$$Au(x) = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left[a(x) \frac{\partial u(x)}{\partial x_j} \right] = f(x), \quad x \in \Omega.$$

Here $f(x)$ and $\bar{u}(x)$ are appointed functions and $u(x)$ is the unknown function. The parametrix $P(x, y)$ is given by

$$P(x, y) = \frac{\ln |x - y|}{2\pi a(y)}, \quad x, y \in \mathbb{R}^2,$$

which meet

$$A_x P(x, y) = \delta(x, y) + R(x, y),$$

where $\delta(x, y)$ is the Dirac delta function and $R(x, y)$ is the remainder given by

$$R(x, y) = \frac{1}{2\pi a(y)} \sum_{i=1}^2 \frac{x_i - y_i}{r} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2.$$

Here r is the radius such that

$$r = |x - y| = \sqrt{(x_i - y_i)(x_i - y_i)}.$$

We also denote that

$$\begin{aligned} Tu(x) &= \sum_{j=1}^2 a(x) \nu_j(x) \frac{\partial u(x)}{\partial x_j}, \quad T_x P(x, y) = \sum_{j=1}^2 a(x) \nu_j(x) \frac{\partial P(x, y)}{\partial x_j} \\ &= \sum_{j=1}^2 \frac{a(x) \nu_j(x) (x_j - y_j)}{2\pi a(y) r^2}, \end{aligned}$$

where $\nu(x) = (\nu_1(x), \nu_2(x))$ is the normal pointing outwards with respect to Ω .

The direct united BDIDE related to Dirichlet problem is given below. See e.g [11] and [12].

$$\begin{aligned}
& c(y)u(y) + \int_{\Omega} R(x, y)u(x) \, d\Omega(x) + \int_{\partial\Omega} P(x, y)Tu(x) \, d\Gamma(x) \\
& = \int_{\partial\Omega} \bar{u}(x)T_x P(x, y) \, d\Gamma(x) + \int_{\Omega} P(x, y)f(x) \, d\Omega(x), \quad y \in \bar{\Omega},
\end{aligned} \tag{1}$$

where $c(y)$ depends on the position of point y i.e.

$$c(y) = \begin{cases} 1 & \text{if } y \in \Omega, \\ 0 & \text{if } y \in \mathbb{R}^2 \setminus \bar{\Omega}, \\ \alpha(y)/2\pi & \text{if } y \in \partial\Omega, \end{cases}$$

and $\bar{\Omega} = \Omega \cup \partial\Omega$ while $\alpha(y)$ is an interior angle at a corner point y of $\partial\Omega$. For $\partial\Omega$ a smooth boundary, it yields that $c(y) = 1/2$.

3. The discretization of BDIDE with the collocation points $x^i \in \bar{\Omega}$

We apply the same interpolation as describe in [11] to equation (1) and taking the collocation point x^i for $x^i \in \bar{\Omega}$ at all J nodes of the mesh elements. We then prevail the following system of J equations for J unknowns $u(x^j)$.

$$c(x^i)u(x^i) + \sum_{x^j \in \bar{\Omega}} K_{ij}^D u(x^j) = Q_i^D + D_i^D, \quad x^i \in \bar{\Omega}, \tag{2}$$

where K_{ij}^D , Q_i^D and D_i^D are defined below.

$$K_{ij}^D = \sum_{m=1}^M \int_{\Omega_m} \phi_j(x) R(x, x^i) \, d\Omega(x) + \sum_{l=1}^L \int_{\partial\Omega_l} P(x, x^i) \left[a(x) \left(\frac{\partial \phi_j(x)}{\partial \nu(x)} \right) \right] \, d\Gamma(x), \tag{3}$$

$$Q_{ij}^D = \sum_{l=1}^L \int_{\partial\Omega_l} \bar{u}(x) T_x P(x, x^i) \, d\Gamma(x), \tag{4}$$

$$D_{ij}^D = \sum_{m=1}^M \int_{\Omega_m} P(x, x^i) f(x) \, d\Omega(x). \tag{5}$$

Here $\phi_j(x)$ are the global shape functions, $\partial\Omega_l$ is the linear iso-parametric element such that $\partial\Omega = \bigcup_{l=1}^L \partial\Omega_l$, $\partial\Omega_k \cap \partial\Omega_m = \emptyset$, $k \neq m$, and Ω_m is the bilinear

quadrilateral mesh element such that $\Omega = \bigcup_{m=1}^M \Omega_m$, $\Omega_k \cap \Omega_m = \emptyset$, $k \neq m$.

In different notations, we can also write equation (3) as the following equation.

$$K_{ij}^D = \sum_{x^j \in \bar{\Omega}_m} \int_{\Omega_m} \phi_j(x) R(x, x^i) \, d\Omega(x) + \sum_{\partial\Omega_l \subset \{\bar{\Omega}_m : x^j \in \bar{\Omega}_m\}} \int_{\partial\Omega_l} P(x, x^i) \left[a(x) \left(\frac{\partial \phi_j(x)}{\partial \nu(x)} \right) \right] \, d\Gamma(x).$$

Suppose that we denote $(\xi_1, \xi_2) =: \xi$ be the intrinsic coordinate on the reference square element with $-1 \leq \xi_1 \leq 1$, $-1 \leq \xi_2 \leq 1$ and η be the intrinsic coordinate on the reference segment with $-1 \leq \eta \leq 1$.

The local two-dimensional shape functions $\Phi_N(\xi)$, $N=1,2,\dots,4$, is given as follows:

$$\begin{aligned}\Phi_1(\xi) &= (1-\xi_1)(1-\xi_2)/4, \quad \Phi_2(\xi) = (1+\xi_1)(1-\xi_2)/4, \\ \Phi_3(\xi) &= (1+\xi_1)(1+\xi_2)/4, \quad \Phi_4(\xi) = (1-\xi_1)(1+\xi_2)/4.\end{aligned}$$

We denote $\Psi_n(\eta)$ as the local one-dimensional shape functions which is given below.

$$\Psi_1(\eta) = \frac{1}{2}(1-\eta), \quad \Psi_2(\eta) = \frac{1}{2}(1+\eta), \quad -1 \leq \eta \leq 1.$$

Moreover, J_{l1} and J_{m2} are denoted for the Jacobians for the transformation in relations in (6) and (7), respectively,

$$x(\xi) = \sum_{N=1}^4 \Phi_N(\xi) X^{mN}, \quad (6)$$

$$x(\eta) = \sum_{n=1}^2 \Psi_n(\eta) X^{ln}, \quad (7)$$

where X^{mN} , $N=1,\dots,4$ is the N th vertex for each quadrilateral domain element Ω_m and X^{ln} , $n=1,2$ is the n th endpoint for each line segment $\partial\Omega_l$.

Denoting $G_{N,i}^m$, H_i^m , $A_{N,i}^l$ and F_i^l as in (8)-(11) below,

$$G_{Ni}^m = \int_{-1}^1 \int_{-1}^1 \Phi_N(\xi) R(x^i, x(\xi)) J_{m2}(\xi) d\xi_1 d\xi_2, \quad (8)$$

$$H_i^m = \int_{-1}^1 \int_{-1}^1 P(x^i, x(\xi)) f(x(\xi)) J_{m2}(\xi) d\xi_1 d\xi_2, \quad (9)$$

$$A_{N,i}^l = \int_{-1}^1 P(x(\eta), x^i) \left[a(x(\eta)) \left(\sum_{p=1}^2 \sum_{k=1}^2 \frac{\partial \Phi_N(\xi)}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_p} \bigg|_{\xi=\xi(\eta)} \nu_p(x(\eta)) \right) \right] J_{l1}(\eta) d\eta, \quad (10)$$

$$F_i^l = \int_{-1}^1 \bar{u}(x(\eta)) T_x P(x(\eta), x^i) J_{l1}(\eta) d\eta, \quad (11)$$

enable us to write (3)-(5) as the following equations.

$$K_{ij}^D = \sum_{x_j \in \Omega_m} G_{N(j,m),i}^m + \sum_{\partial\Omega_l \subset \{\bar{\Omega}_{2m}: x^j \in \bar{\Omega}_{2m}\}} A_{N(j,m),i}^l, \quad (12)$$

$$Q_i^D = \sum_{l=1}^L F_i^l, \quad (13)$$

$$D_i^D = \sum_{m=1}^M H_i^m. \quad (14)$$

Here $N(j, m)$ denotes the local number of the node x^j on each domain element Ω_m .

All the integrals in (8)-(11) are evaluated by Gauss-Laguerre quadrature formula. However, the integrals in (8)-(11) pose singularity when $x = y$. We employ Duffy transformation to handle the singularity for the domain integrals in (8) and (9). The semi-analytic method as proposed in [12] for Dirichlet BDIDE is used to evaluate the boundary integral that consists of logarithmic singularity in (10). For boundary integral (11), its singularity is understood in the Cauchy principal value sense. See e.g. [1].

The discretization of BDIDE with the collocation points $x^i \in \Omega$

Rather than having BDIDE (1) been applied at $y \in \partial\Omega$, we can also substitute Dirichlet boundary condition $u = \bar{u}$ on $\partial\Omega$. The idea is rather than we choose the collocation point x^i for $x^i \in \bar{\Omega}$ at all J nodes of the mesh during the interpolation process, we can actually choose the collocation point x^i at $J - J_D$ nodes of the mesh i.e for $x^i \in \Omega$. Here J_D denotes the number of boundary nodes of the mesh. Hence, we can segregate $\sum_{x^j \in \Omega} K_{ij}^D u(x^j)$ in (2) to two parts i.e.

$$\sum_{x^j \in \Omega} K_{ij}^D u(x^j) = \sum_{x^j \in \Omega} K_{ij}^D u(x^j) + \sum_{x^j \in \partial\Omega} K_{ij}^D \bar{u}(x^j).$$

The second part $\sum_{x^j \in \partial\Omega} K_{ij}^D \bar{u}(x^j)$ can be shifted to the right-hand side of (2).

Hence, (2) yields the system of $J - J_D$ linear algebraic equations for $J - J_D$ unknowns, given below.

$$u(x^i) + \sum_{x^j \in \Omega} K_{ij}^D u(x^j) = - \sum_{x^j \in \partial\Omega} K_{ij}^D \bar{u}(x^j) + Q_i^D + D_i^D, \quad x^i \in \Omega, \quad (15)$$

Here K_{ij}^D , Q_i^D , and D_i^D are as (12)-(14). In this paper, the system of equations (15) is solved by LU decomposition method.

4. Numerical results

In the numerical computations, we use Fortran (Intel Visual Fortran Compiler Professional Edition 11.1) with double precision accuracy. We calculate the relative errors for the estimate solution and their gradients as given in formulae below.

$$\epsilon(u) = \frac{\max_{1 \leq j \leq J} |u_{approx}(x^j) - u_{exact}(x^j)|}{\max_{1 \leq j \leq J} |u_{exact}(x^j)|}, \quad \epsilon(\nabla u) = \frac{\max_{1 \leq j \leq J} |\nabla u_{approx}(x_c^m) - \nabla u_{exact}(x_c^m)|}{\max_{1 \leq j \leq J} |\nabla u_{exact}(x_c^m)|}.$$

Here x_c^m represents the centers of the mesh elements Ω_m . The test domains that are used are a square $1 < x_1 < 2, 1 < x_2 < 2$, a circular domain with a unit radius and centered at coordinate (2, 2) and a parallelogram with vertices (3, 1), (4, 1), (6, 2) and (5, 2). These three test domains are also the test domains used in [3] for solving Neumann BDIE. The following two Dirichlet boundary value problems are taken into account in our numerical experiments:

Test 1: $a(x) = x_2, f(x) = 0$ for $x \in \Omega \cup \partial\Omega$ and $\bar{u}(x) = x_1$ for $x \in \partial\Omega$,

Test 2: $a(x) = x_2^2, f(x) = 2x_2^2$ for $x \in \Omega \cup \partial\Omega$ and $\bar{u}(x) = x_1^2$ for $x \in \partial\Omega$.

For each domain, square, circular and parallelogram, we compute the comparative results of the relative error for both the estimated solutions u_{approx} and their gradient ∇u_{approx} against the number of nodes J . The results are shown in Figures 1-3.

As in [8] and [9], we let the error $\epsilon(u)$ depends on the number of nodes J as well as the average of the linear size for the elements, h such that $\epsilon(u) \approx J^{-q/2} \approx h^q$.

Based on our experiments, the rate of the convergence for the solutions of the Dirichlet BDIDE as J increases is near to the results obtained for BDIE for Neumann problem presented in [9]. It gives linear and quadratic convergence i.e. $q \approx 1$ in Test 1 and $q \approx 2$ in Test 2.

The precision in Test 2 is lower than Test 1 since unlike Test 1, not only numerical integral approximation is concerned for Test 2 but also the computations of the piece-wise bilinear interpolation for the quadratic exact solution $\bar{u}(x) = x_1^2$ contributes to the total of the error as compared to the linear exact solution $\bar{u}(x) = x_1$ that only involves integration error.

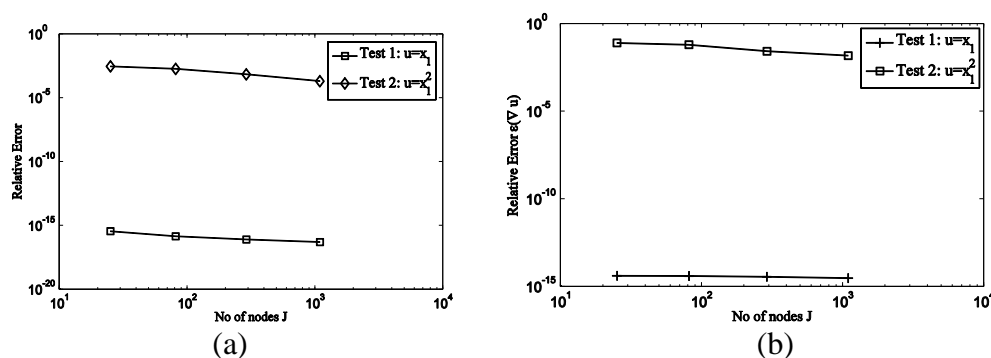


Figure 1. Relative errors of the estimate solutions (a) and their gradients (b), on the square against number of nodes J

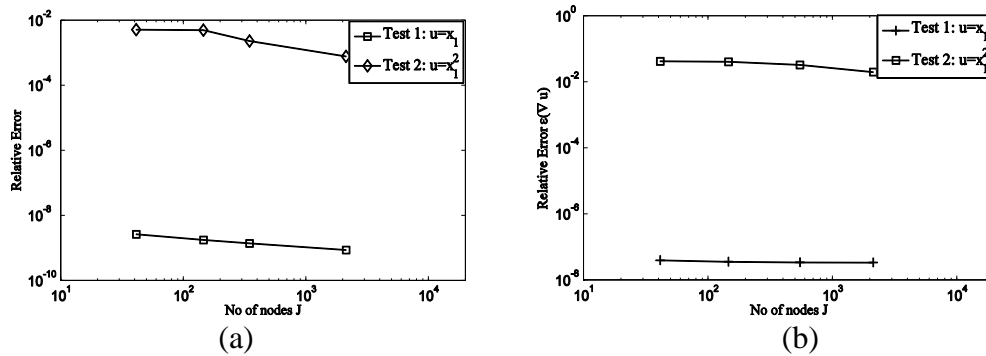


Figure 2. Relative errors of the estimate solutions (a) and their gradients (b), on circular test domain against number of nodes J

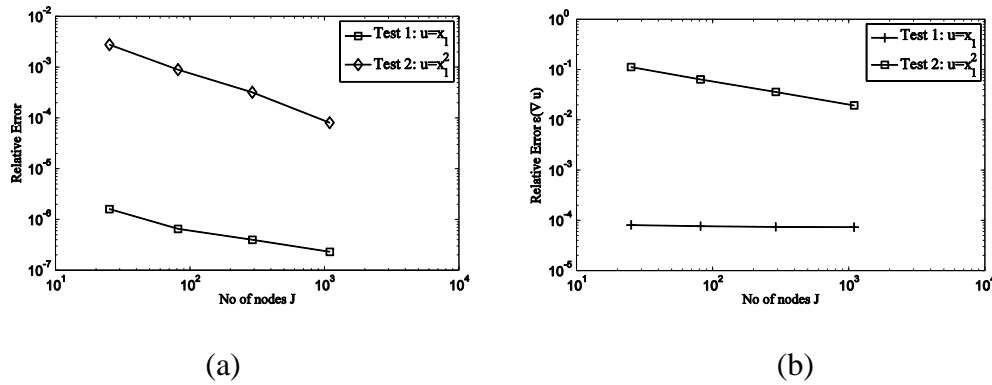


Figure 3. Relative errors of the estimate solutions (a) and their gradients (b), on parallelogram against number of nodes J

5. Conclusions

We have shown that there are two systems of linear systems can be obtained from the Dirichlet BDIDE. We can also say that there are two different approaches during the interpolation process. The first approach is by choosing the collocation points x^i for $x^i \in \bar{\Omega}$ i.e for all J nodes like used in [7] for numerical solution of BDIE associated to Neumann problem. The second approach is by choosing the collocation points x^i with less no of nodes points i.e. at $J - J_D$ nodes during the interpolation process such that it is taken for only $x^i \in \Omega$. In this paper, we show the numerical solution for Dirichlet BDIDE of the second way. The second approach seems to be an advantage in respect of numerical experiment less collocation points would lead to less computation's time and less effort. However, the spectral properties of the matrix operator obtained from the system of equations is still need to be investigated. This is essential as to ensure that the system of equations obtained from the discretized Dirichlet BDIDE with less number

of collocation points not only can be solved by a direct method but also any iterative method.

We solved the algebraic system of linear equations by employing a direct method i.e. by making use of LU decomposition method. The results of our numerical experiments on several test domains for the considered linear and quadratic solution Dirichlet problems are acceptable. For linear solution Dirichlet problem on square domain, the obtained relative errors are up to 10^{-14} which is considered as high accuracy for a double precision numerical code.

The accuracy in linear Dirichlet problem (Test 1) is much higher when we compare to the quadratic Dirichlet problem (Test 2) since there is only integral operator approximation error is occurred. But, for quadratic Dirichlet problem (Test 2), other than integration error, the quadratic exact solution gives the piece-wise bi-linear interpolation to contribute most of the whole error.

From the numerical experiments, we also validate that the semi-analytic method for the Dirichlet BDIDE as proposed in [12] give high accuracy results. In [12], only theoretical works have been presented and no numerical experiments is shown.

It is also interesting to do the numerical experiment for the discretized BDIDE by taking x^j at all J nodes to compare the accuracy with the results obtained in this paper for BDIDE with less number of collocation points. Even though in terms of computational time it is not difficult to predict that discretized BDIDE with less number of collocation points requires less computational time, the accuracy of the results yield from both approaches are still need to be investigated.

In this paper, we also use the semi-analytic integration method as proposed in [12] to handle the integration of the kernel that involves singularity of logarithmic function as occurs in Dirichlet BDIDE. The paper [12] provides the theoretical works on semi-analytic integration method and no numerical validation is presented in the paper.

From the numerical experiments that have been done in this paper, we validate that the use of the semi-analytic integration method as first proposed in [12] does give high accuracy results for the Dirichlet BDIDE.

This semi-analytic integration method is predicted to reduce the integration error. Hence, it is more prominent for the result in test 1 since its Dirichlet BDIDE has linear exact solution $\bar{u}(x) = x_1$ that deals only with integration error. Whereas, for the quadratic solution $\bar{u}(x) = x_1^2$ as in test 2, the piece-wise interpolation error contributes to the total error.

Acknowledgements. The author is thankful to Prof. Sergey Mikhailov, Department of Mathematics, Brunel University, London for the suggestions throughout the research.

The author would also like to thank Universiti Pendidikan Sultan Idris (UPSI) and Ministry of Higher Education in Malaysia with the RAGS code: 2014-0122-101-72 for their financial contribution in respect for this study.

References

- [1] G. Beer, *Programming the Boundary Element Method*, West Sussex: John Wiley & Wiley, 2001, ch. 6, 119-130.
- [2] O. Chkadua, S.E. Mikhailov and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, I: Equivalence and invertibility, *Journal of Integral Equations and Applications*, **21** (2009), 499-543. <http://dx.doi.org/10.1216/jie-2009-21-4-499>
- [3] O. Chkadua, S.E. Mikhailov and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient. Part II. Solution regularity and asymptotics, *J. Integral Equations and Appl.*, **22** (2010), no. 1, 19-37. <http://dx.doi.org/10.1216/jie-2010-22-1-19>
- [4] D. Hilbert, *Grundzüge Einer Allgemeinen Theorie der Linearen Integralgleichungen*, Teubner, Leipzig, 1912.
- [5] J.T. Katsikadelis, *Boundary Elements Theory and Applications*, Elsevier, Oxford, 2002.
- [6] E.E. Levi, I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali, *Mem. Soc. Ital. dei Sc. XL*, 16:1–112, 1909.
- [7] S.E. Mikhailov, Analysis of united boundary-domain integro-differential and integral equations for a mixed BVP with variable coefficient, *Math. Methods Appl. Sci.*, **29** (2006), no. 6, 715-739. <http://dx.doi.org/10.1002/mma.706>
- [8] S.E. Mikhailov and N.A. Mohamed, Iterative solution of boundary-domain integral equation for {BVP} with variable coefficient, *Proceedings of the 8th UK Conference on Boundary Integral Methods* (edited by D. Lesnic), Leeds University Press, UK, 2011, 127-134.
- [9] S.E. Mikhailov and N.A. Mohamed, Numerical solution and spectrum of boundary-domain integral equation for the Neumann BVP with a variable coefficient, *International Journal of Computer Mathematics*, **89** (2012), 1488-1503. <http://dx.doi.org/10.1080/00207160.2012.679733>

- [10] C. Miranda, *Partial Differential Equations of Elliptic Type*, Second revised edition, Springer-Verlag, Berlin, 1970.
<http://dx.doi.org/10.1007/978-3-662-35147-5>
- [11] N.A. Mohamed, *Numerical Solution and Spectrum of Boundary-Domain Integral Equations*, Ph.D. Thesis, Brunel University, London, 2013.
- [12] N.A. Mohamed, Semi-Analytic Integration Method for Direct United Boundary-Domain Integro-Differential Equation Related to Dirichlet Problem, *International Journal of Applied Physics and Mathematics*, **4** (2014), no. 3, 149-154. <http://dx.doi.org/10.7763/ijapm.2014.v4.273>
- [13] F. Paris and J. Cañas, *Boundary Element Method Fundamentals and Applications*, Oxford University Press, Oxford, 1997.
- [14] A. Pomp, *The Boundary-domain Integral Method for Elliptic Systems*, Lect. Notes in Math., Vol. 1683, Springer-Verlag, Berlin-Heidelberg, 1998.
<http://dx.doi.org/10.1007/bfb0094576>

Received: March 19, 2016; Published: August 11, 2016