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Nur Syaza Mohd Yusop and Nurul Akmal Mohamed

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The System of Equations for Mixed BVP With One Dirichlet Boundary Condition and Three Neumann Boundary Conditions

Nur Syaza Mohd Yusop[1, a)] and Nurul Akmal Mohamed[1, b)]

[1]*Department of Mathematics, Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris 35900 Tanjong Malim, Perak*

[a)]syaza_919931@yahoo.com
[b)]akmal.mohamed@fsmt.upsi.edu.my

Abstract. Boundary Element Method (BEM) is a numerical way to approximate the solutions of a Boundary Value Problem (BVP). The potential problem which involves the Laplace's equation on the square shape domain will be considered where the boundary is divided into four sets of linear boundary elements. We study the derivation system of equation for mixed BVP with one Dirichlet Boundary Condition (BC) is prescribed on one element of the boundary and Neumann BC on the other three elements. The mixed BVP will be reduced to a Boundary Integral Equation (BIE) by using a direct method which involves Green's second identity representation formula. Then, linear interpolation is used where the boundary will be discretized into some linear elements. As the result, we then obtain the system of linear equations. In conclusion, the specific element in the mixed BVP will have the specific prescribe value depends on the type of boundary condition. For Dirichlet BC, it has only one value at each node but for the Neumann BC, there will be different values at the corner nodes due to outward normal. Therefore, the assembly process for the system of equations related to the mixed BVP may not be as straight forward as Dirichlet BVP and Neumann BVP. For the future research, we will consider the different shape domains for mixed BVP with different prescribed boundary conditions.

INTRODUCTION

The BVP of a Partial Differential Equation (PDE) can be reduced to BIE by using a direct method which involves Green's second identity representation formula. There are two ways that can be applied in solving BIE which are by analytically and numerically. However, analytical method cannot handle all BIE. Therefore, numerical method is an alternative to get the approximate solution for BIE [1]. The common method used is Finite Element Method (FEM). But, there is another method that can be considered, which is a Boundary Element Method (BEM).

In some cases of BVPs such as those involving infinite domains or 3-dimensional complicated domains, the BEM is more preferable method in solving BVP as compared to FEM. This is due to the discretization of the problem domain only involves the discretization of boundary. Another advantage is the BEM reduces the problem dimensionality by one. Therefore, we will not take too long time in solving the problem because of the reduction in dimensionality. Although, BEM is less time consuming process, but this method has difficulty when dealing with integrals that involve singular integrands [2].

In this paper, only the potential problem which involves the Laplace's equation on the square shape domain will be discussed where the boundary is divided into four sets of linear boundary elements. The aim of this study is to

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derive a system of equations relating mixed BVP of PDE i.e combination of Dirichlet BC on one element of the boundary and Neumann BC on the other three elements.

Representation Equation for Mixed BVP

In this work, we illustrate the process of BEM using potential problem which involve the Laplace's equation. Let u be a potential function which defined in the domain Ω with boundary $\Gamma := \partial\Omega$, and outward normal ν. The function u is required to satisfy the following PDE which is known as Laplace's equation.

$$Lu(x) := \nabla^2 u(x) = 0\,, \qquad x \in \Omega\,. \tag{1}$$

The mixed boundary conditions corresponding to this problem are the combination of two types which are Dirichlet BC $u(x) = \bar{u}(x)$ on $\partial\Omega_D$ and Neumann BC $t(x) = a(x)\dfrac{\partial u(x)}{\partial \boldsymbol{\nu}} = \bar{t}(x)$ on $\partial\Omega_N$ where $\partial\Omega_D \cap \partial\Omega_N = \varnothing$ and $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$ .The first step that we have to carry out in the BEM is transforming the PDE to the boundary integral equation through the direct way by using Green's theorem as mentioned in [3]. Hence, equation (1) can be written as Green's representation formula i.e.

$$u(y) = \int_{\Gamma} \big(t(x)U(x,y) - u(x)T(x,y)\big)\, d\Gamma\,. \tag{2}$$

The kernels $U(x,y)$ and $T(x,y)$ are given as follows.

$$U(x,y) = -\frac{1}{2\pi}\ln|x-y| = -\frac{1}{2\pi}\ln\, r, \qquad \text{for} \qquad x,y \in \mathbb{R}^2, \tag{3}$$

$$T(x,y) = \frac{\partial U(x,y)}{\partial \boldsymbol{\nu}} = \nu_{x_1}\frac{\partial U(x,y)}{\partial x_1} + \nu_{x_2}\frac{\partial U(x,y)}{\partial x_2}, \qquad \text{for} \qquad x,y \in \mathbb{R}^2, \tag{4}$$

where r is denoted as Euclidean distance. However, the equation (2) only exists when load point y approaches field points x without considering the situation of singularity such that points y coincides with point x . Therefore, limiting process is carried out for the situation when points y and x coincide. As described in [6], equation (2) comes out with a new term $c(y)$ and we can obtain the BIE as given below.

$$c(y)u(y) = \int_{\Gamma} t(x)U(x,y)d\Gamma(x) - \int_{\Gamma} u(x)T(x,y)d\Gamma(x) \tag{5}$$

where $c(y) = \dfrac{\alpha}{2\pi}$ and α is an interior angle at a corner point of the boundary. In our considered square shape domain, the value of α at a corner point y of the boundary $\partial\Omega$ is 90 degree, then we have $c(y) = \dfrac{1}{4}$.

Discretization of BIE for mixed BVP

In the case of mixed BVP, we can rearrange the integrals in equation (5). So that, the integral with the terms involving unknown BC value are placed to the Left Hand Side (LHS) of the equation, whereas the integral with the known prescribed BC values are placed to the Right Hand Side (RHS) of the equation. Therefore, the rearrangement of equation (5) for mixed BVP can be rewritten as follows

$$
\begin{aligned}
c(y)u(y) + \int_{\Gamma_N} u(x)T(x,y)\, d\Gamma(x) - \int_{\Gamma_D} t(x)U(x,y)\, d\Gamma(x) = \\
\int_{\Gamma_N} \bar{t}(x)U(x,y)\, d\Gamma(x) - \int_{\Gamma_D} \bar{u}(x)T(x,y)\, d\Gamma(x).
\end{aligned} \tag{6}
$$

Equation (6) can be solved by numerical method. In two dimensional case, the boundary of the surface is discretized into E linear segments of straight lines such that $\partial\Omega \simeq \partial\Omega_1 \cup \partial\Omega_2 \cup \ldots \cup \partial\Omega_E$. The nodes are located at the ends of linear elements and the shape functions have to be introduced. Next, we interpolate boundary values in between the nodes of linear elements such that

$$
u(x) = \sum_{j=1}^{J} \phi_j(x)\, u(x^j) \qquad\qquad x, x^j \in \partial\Omega \tag{7}
$$

where $\phi_j(x)$ is the global shape function, x^j is the global nodes and J is the number of nodes. By applying the interpolation equation (7) to equation (6) and placing the collocation point x^i at all global nodes, we get the system of equations as follows.

$$
c(x^i)u(x^i) + \sum_{x^j \in \Gamma_N} K_{ij} u(x^j) - \sum_{x^j \in \Gamma_D} G_{ij} t(x^j) = \sum_{x^j \in \Gamma_N} \overline{G_{ij}}\ \bar{t}(x^j) - \sum_{x^j \in \Gamma_D} \overline{K_{ij}}\ \bar{u}(x^j)\ ,\quad x^i \in \partial\Omega, \tag{8}
$$

where K_{ij} , G_{ij} , $\overline{K_{ij}}$ and $\overline{G_{ij}}$ are read as:

$$
K_{ij} = \int_{\Gamma_N} \phi_j(x) T(x^i, x) d\Gamma(x)\ , \tag{9}
$$

$$
G_{ij} = \int_{\Gamma_D} \phi_j(x) U(x^i, x) d\Gamma(x)\ , \tag{10}
$$

$$
\overline{K_{ij}} = \int_{\Gamma_D} \phi_j(x) T(x^i, x) d\Gamma(x)\ , \tag{11}
$$

$$
\overline{G_{ij}} = \int_{\Gamma_N} \phi_j(x) U(x^i, x) d\Gamma(x). \tag{12}
$$

The term E represents the set consisting of element number where $\mathsf{E} = \{e \in \mathsf{E}\} = \{1, 2, \ldots, E\}$. Suppose D and N be subsets of E i.e. $\mathsf{D} \subseteq \mathsf{E}$ and $\mathsf{N} \subseteq \mathsf{E}$. Then, we have $\mathsf{D} \cap \mathsf{N} = \phi$ where D is a set of element numbers with known Dirichlet BC, $\mathsf{D} = \{e \in \mathsf{E} \mid e \text{ is an element with Dirichlet BC prescribe value}\}$ and N represents as the set of element numbers with known Neumann BC, $\mathsf{N} = \{e \in \mathsf{E} \mid e \text{ is an element with Neumann BC prescribe value}\}$. The element numbers on the boundary Γ_D and Γ_N are specified in the set notation above. Therefore, we obtain

$$
K_{ij} = \sum_{\mathsf{N}} \int_{\Gamma_{\mathsf{N}(e)}} \phi_j(x) T(x^i, y) d\Gamma(x), \tag{13}
$$

$$
G_{ij} = \sum_{\mathsf{D}} \int_{\Gamma_{\mathsf{D}(e)}} \phi_j(x) U(x^i, y) d\Gamma(x), \tag{14}
$$

$$
\overline{K_{ij}} = \sum_{\mathsf{D}} \int_{\Gamma_{\mathsf{D}(e)}} \phi_j(x) T(x^i, y) d\Gamma(x), \tag{15}
$$

$$
\overline{G_{ij}} = \sum_{\mathsf{N}} \int_{\Gamma_{\mathsf{N}(e)}} \phi_j(x) U(x^i, y) d\Gamma(x), \tag{16}
$$

In the remaining parts, we write the equation (13)-(16) to the local node for the numerical purpose. Therefore, the relationships between local nodes and global nodes are introduced as $x^j = x^{j(e,n)} = x_n^e$, for $x_n^e \in \partial\Omega_e$ that implies $u(x^j) = u(x_n^e)$. Then, it can be verified in [4] and [5] that the Cartesian coordinates of a point on element with the intrinsic coordinate η are given by

$$\begin{Bmatrix} x_1(\eta) \\ x_2(\eta \end{Bmatrix} = \sum_{n=1}^{2} \psi_n(\eta) \begin{Bmatrix} x_{1n}^e \\ x_{2n}^e \end{Bmatrix} \ , \ -1 \le \eta \le 1 \ , \tag{17}$$

where $\psi_n(\eta)$ are element shape functions for linear segment which are given as

$$\psi_1(\eta) = \frac{1}{2}(1-\eta) ,$$
$$\psi_2(\eta) = \frac{1}{2}(1+\eta)$$

where $-1 \le \eta \le 1$ .Therefore, equations (13)-(16) are rewritten as

$$K_{ij} = \sum_{x^j \in \Gamma_{N(e)}} A_{ni}^e ,$$
$$G_{ij} = \sum_{x^j \in \Gamma_{D(e)}} F_{ni}^e ,$$
$$\overline{K_{ij}} = \sum_{x^j \in \Gamma_{D(e)}} A_{ni}^e ,$$
$$\overline{G_{ij}} = \sum_{x^j \in \Gamma_{N(e)}} F_{ni}^e$$

where the integral A and integral F can be interpreted as

$$A_{ni}^e = \int_{-1}^{1} \psi_n(\eta) \ T(x^i,\eta) J_{e1}(\eta) d\eta, \tag{18}$$

$$F_{ni}^e = \int_{-1}^{1} \psi_n(\eta) \ U(x^i,\eta) J_{e1}(\eta) d\eta. \tag{19}$$

The term n represents local number i.e. $n = 1,2$ for linear element and J_{e1} in equation (18)-(19) represents Jacobian where $J_{e1} = \dfrac{|\partial\Omega_e|}{2}$ and $|\partial\Omega_e|$ is length of each boundary element [5].

Description of Problem 1

In this paper, we consider a simple problem 1 by using the surface of square shape with three prescribed values for Neumann BC and one prescribed value for Dirichlet BC. This problem 1 is shown in Figure 1.

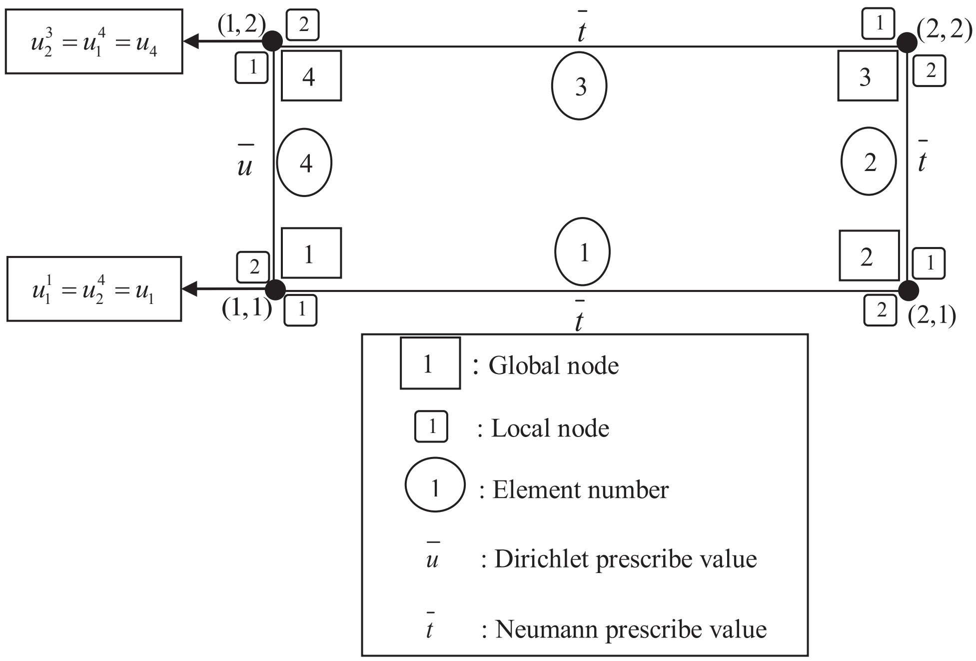

Figure 1. The considered domain with mixed BC

The integrals terms in equation (8) for the mixed BVP are associated with known values and unknown values for both Dirichlet and Neumann boundary values. By referring to the problem in Figure 1, $\bar{u}$ is prescribed on one element of the boundary and $\bar{t}$ on the other three elements of the boundary. Therefore, the classification for integrals of known values and unknown values are shown in Table 1 as below

TABLE 1. Integrals of known and unknown values in each elements and nodes for Problem 1

	Element 1		**Element 2**		**Element 3**		**Element 4**	
Local nodes	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
Known on RHS	$F_{1i}^1\bar{t}_1^1$	$F_{2i}^1\bar{t}_2^1$	$F_{1i}^2\bar{t}_1^2$	$F_{2i}^2\bar{t}_2^2$	$F_{1i}^3\bar{t}_1^3$	$F_{2i}^3\bar{t}_2^3$	$A_{1i}^4\bar{u}_1^4$	$A_{2i}^4\bar{u}_2^4$
Unknown on LHS	$A_{1i}^1u_1^1$	$A_{2i}^1u_2^1$	$A_{1i}^2u_1^2$	$A_{2i}^2u_2^2$	$A_{1i}^3u_1^3$	$A_{2i}^3u_2^3$	$F_{1i}^4t_1^4$	$F_{2i}^4t_2^4$

Unlike Neumann BC values that takes two different values at each corner node due to outward normal, the Dirichlet BC value only takes one value at each node [6]. In this case, there is no problem with the corner nodes 1 and 4. Hence, Neumann BC value is unknown at nodes 1 and 4, whereas at boundary nodes 2 and 3, the Dirichlet BC value is unknown. Therefore, the node $u_1^1 = u_2^4 = u_1$ and $u_2^3 = u_1^4 = u_4$. Then, inserting the terms in the Table 1 accordingly in equation (15) leads to

$$\begin{aligned} c(x_i)u(x_i) + \left[A_{2i}^1 + A_{1i}^2\right]u_2 + \left[A_{2i}^2 + A_{1i}^3\right]u_3 - F_{1i}^4t_1^4 - F_{2i}^4t_2^4 = F_{1i}^1\bar{t}_1^1 + F_{2i}^1\bar{t}_2^1 \\ + F_{1i}^2\bar{t}_1^2 + F_{2i}^2\bar{t}_2^2 + F_{1i}^3\bar{t}_1^3 + F_{2i}^3\bar{t}_2^3 - \left[A_{2i}^4 + A_{1i}^1\right]u_1 - \left[A_{1i}^4 + A_{2i}^3\right]u_4. \end{aligned} \tag{27}$$

By using collocation point from equation (27), we obtain the following system of equations in the matrix form $B\boldsymbol{x} = H$. Matrix B is a $j \times j$ matrix where matrix rows and columns are representation of collocation points and global nodes, respectively. Vector $\boldsymbol{x}$ is $j\times 1$ and H is $j\times 1$ where j represent the number of global nodes. Therefore, equation (27) yields the matrix below

$$\begin{bmatrix} -F_{21}^{4} & \left(A_{21}^{1}+A_{11}^{2}\right) & \left(A_{21}^{2}+A_{11}^{3}\right) & -F_{11}^{4} \\ -F_{22}^{4} & \left(A_{22}^{1}+A_{12}^{2}+\dfrac{1}{4}\right) & \left(A_{22}^{2}+A_{12}^{3}\right) & -F_{12}^{4} \\ -F_{23}^{4} & \left(A_{23}^{1}+A_{13}^{2}\right) & \left(A_{23}^{2}+A_{13}^{3}+\dfrac{1}{4}\right) & -F_{13}^{4} \\ -F_{24}^{4} & \left(A_{24}^{1}+A_{14}^{2}\right) & \left(A_{24}^{2}+A_{14}^{3}\right) & -F_{14}^{4} \end{bmatrix} \begin{bmatrix} t_2^4 \\ u_2 \\ u_3 \\ t_1^4 \end{bmatrix} = H, \tag{28}$$

where H is the RHS 4×1 matrix, $H=\left[H_i, i=1,2,3,4\right]$.

$$H_i = F_{1i}^{1}\bar{t}_1^{1} + F_{2i}^{1}\bar{t}_2^{1} + F_{1i}^{2}\bar{t}_1^{2} + F_{2i}^{2}\bar{t}_2^{2} + F_{1i}^{3}\bar{t}_1^{3} + F_{2i}^{3}\bar{t}_2^{3} - \left[A_{2i}^{4} + A_{1i}^{1} + \frac{1}{4}\right]u_1 - \left[A_{1i}^{4} + A_{2i}^{3} + \frac{1}{4}\right]u_4. \tag{29}$$

This system (29) can then be solved by using direct or iterative methods such as Gauss elimination method to get the approximation values for unknown boundary values.

CONCLUSION

In this paper, we only focus on the theoretical aspect of BEM on a square shape domain. The boundary is discretized into four set of linear boundary elements. We only consider a simple mixed BVP as in Figure 1 where only one Dirichlet BC is prescribed and Neumann BCs on the other three elements. This considered mixed BVP leds to a square system of equations and can be expressed as a matrix system. The matrix system obtained in equation (28) can be solved by direct or iterative method. However, not all mixed BVPs contributes to the square matrix operator and that will be our future research. In the future research, we are interested to extend with the numerical method and approximate the solution for mixed BVP.

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