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Spectrum of Dirichlet BDIDE operator

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ABSTRACT

In this paper, we present the distribution of some maximal eigenvalues that are obtained numerically from the discrete Dirichlet Boundary Domain Integro-Differential Equation (BDIDE) operator. We also discuss the convergence of the discrete Dirichlet BDIDE that corresponds with the obtained absolute value of the largest eigenvalues of the discrete BDIDE operator. There are three test domains that are considered in this paper, i.e., a square, a circle, and a parallelogram. In our numerical test, the eigenvalues disperse as the power of the variable coefficient increases. Not only that, we also note that the dispersion of the eigenvalues corresponds with the characteristic size of the test domains. It enables us to predict the convergence of an iterative method. This is an advantage as it enables the use of an iterative method in solving Dirichlet BDIDE as an alternative to the direct methods.

Keywords: Direct united boundary-domain integro-differential equation, Dirichlet problem, partial differential equation, Green's identity and eigenvalues.

1. Introduction

We can describe the physical phenomena such as wave propagation, diffusion (heat) and fluid-dynamics phenomena by using partial differential equations (PDEs). The PDEs can be solved by numerically and analytically, but the later can only handle simple and less complicated problems. Therefore, the use of numerical methods in solving PDEs is more preferable for the general problems. There are many numerical methods for the PDEs that have been formulated. Amongst the most used numerical method is the volume-discretisation methods e.g. Finite Element Method. Boundary Element Method (BEM) is another numerical method that give equivalent in term of effectiveness to the volume-discretisation methods. The BEM is however exclusively can be used in solving Boundary Value Problems (BVPs) for the PDE with constant coefficient. The representation formula for the potential theory i.e. Green identity on BVP for PDE with constant coefficient yields integral equations that consists of only boundary integrals. However, for BVP related to PDE with variable coefficient, the Green's identity yields integral equations that consists of not only boundary integrals but also domain integrals. Such integral equation is known as Boundary-Domain integral Equation (BDIE) or Boundary-Domain integro-differential equation (BDIDE). See e.g. Mohamed et al. (2016a) and Mikhalov (2006).

The analysis of the spectral properties for the discrete Neumann BDIE operator has been discussed in Mikhalov and Mohamed (2012). In this paper, we further the discussions of the spectral behaviour to the discrete Dirichlet BDIDE operator. We check the distributions of its eigenvalues which we have obtained numerically. We also discuss the convergence of the discrete Dirichlet BDIDE that corresponds with the obtained absolute value of the largest eigenvalues for the discrete BDIDE operator. We also give a suggestion on how to handle the distribution of the complex eigenvalues which their complex values are exterior to a unit circle whereas their real values stay within a unit circle.

2. Dirichlet BDIDE

We take into account the linear second order elliptic PDE as given below,

$$Au(x) = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left[b(x)\frac{\partial u(x)}{\partial x_j}\right] = g(x), \quad x \in \Omega, \tag{1}$$

where $g(x), \overline{u}(x)$ and $b(x) > 0$ are known functions and $u(x)$ is the unknown function. Let $\hat{P}(x,y)$ be a parametrix of operator A in (1), which is given by,

$$\hat{P}(x,y) = \frac{\ln r}{2\pi b(y)}, \quad x, y \in \mathbb{R}^2. \tag{2}$$

The radius r also satisfies

$$r = |x-y| = \sqrt{(x_i - y_i)(x_i - y_i)}.$$

Equation (2) fulfills equation below,

$$A_x \hat{P}(x,y) = \delta(x,y) + \hat{R}(x,y). \tag{3}$$

Here $\delta(x,y)$ be the Dirac delta function and $\hat{R}(x,y)$ is the remainder, i.e.

$$\hat{R}(x,y) = \frac{1}{2\pi b(y)} \sum_{i=1}^{2} \frac{x_i - y_i}{r} \frac{\partial b(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2. \tag{4}$$

We denote

$$Tu(x) = \sum_{j=1}^{2} b(x)\nu_j(x) \frac{\partial u(x)}{\partial x_j},$$

and

$$T_x \hat{P}(x,y) = \sum_{j=1}^{2} b(x)\nu_j(x) \frac{\partial \hat{P}(x,y)}{\partial x_j} = \sum_{j=1}^{2} \frac{b(x)\nu_j(x)(x_j - y_j)}{2\pi b(y) r^2},$$

where $\nu(x) = (\nu_1(x), \nu_2(x))$ is the outward normal to Ω which is pointing towards the exterior of the boundary $\partial\Omega$.

The second Green's identity is given by

$$\int_{\Omega} [uAv - vAu] \mathrm{d}\Omega = \int_{\partial\Omega} [uTv - vTu] \; \mathrm{d}\Gamma. \tag{5}$$

By supposing v be the parametrix $\hat{P}(x,y)$ as in (2), Mohamed et al. (2016a) and Mohamed et al. (2016b) show that (5) leads to the BDIDE related to Dirichlet problem as given below,

$$\begin{aligned} d(y)u(y) \quad &+ \int_{\Omega} u(x)\hat{R}(x,y)\mathrm{d}\Omega(x) + \int_{\partial\Omega} \hat{P}(x,y)Tu(x) \; \mathrm{d}\Gamma(x) \\ &= \int_{\partial\Omega} T_x \hat{P}(x,y)\overline{u}(x)\mathrm{d}\Gamma(x) + \int_{\Omega} g(x)\hat{P}(x,y) \; \mathrm{d}\Omega(x), \quad y \in \overline{\Omega}, \end{aligned} \tag{6}$$

where $\overline{\Omega} = \Omega \cup \partial\Omega$.

The location of point y has an effect on the coefficient $d(y)$ such that

$$d(y) = \begin{cases} 1 & \text{if} \quad y \in \Omega, \\ 0 & \text{if} \quad y \in \mathbb{R}^2 \backslash \overline{\Omega}, \\ \frac{\beta(y)}{2\pi} & \text{if} \quad y \in \partial\Omega, \end{cases}$$

where the function $\beta(y)$ is an interior angle with respect to the corner point y of $\partial\Omega$.

The angle $\beta(y)$ will take the value π whenever $\partial\Omega$ be a smooth boundary that gives $d(y) = 1/2$. By substituting Dirichlet boundary condition for the out of integral term, equation (6) is equivalent to the equation given below.

$$\begin{aligned} u(y) \quad &+ \int_{\partial\Omega} Tu(x)\hat{P}(x,y)\, \mathrm{d}\Gamma(x) + \int_{\Omega} u(x)\hat{R}(x,y)\mathrm{d}\Omega(x) = \overline{u}(y)\,(1 - d(y)) \\ &+ \int_{\partial\Omega} T_x\hat{P}(x,y)\overline{u}(x)\, \mathrm{d}\Gamma(x) + \int_{\Omega} g(x)\hat{P}(x,y)\, \mathrm{d}\Omega(x), \quad y \in \overline{\Omega}. \end{aligned} \tag{7}$$

Let the solution be sought at J points such that we will have J number of node points $x^j \in \overline{\Omega}$. Applying interpolation, and the collocation points x^i are placed in J element function nodes, we obtain

$$\begin{aligned} u(x^i) \quad &+ \sum_{x^j \in \overline{\Omega}} \hat{K}_{ij} u(x^j) = \left(1 - d(x^i)\right)\overline{u}(x^i) + \hat{Q}_i + \hat{D}_i, \\ & x^i \in \overline{\Omega},\ x^j \in \overline{\Omega},\ j = 1, 2, \ldots, J, \end{aligned} \tag{8}$$

where $\hat{K}_{ij}$, $\hat{Q}_i$ and $\hat{D}_i$ are given below,

$$\begin{aligned}\hat{K}_{ij} &= \int_{\Omega} \eta_j(x)\hat{R}(x,y)\,\mathrm{d}\Omega(x) + \int_{\partial\Omega} b(x)\left(\frac{\partial \eta_j(x)}{\partial \nu(x)}\right)\hat{P}(x,y)\,\mathrm{d}\Gamma(x) \\ &= \sum_{m=1}^{M}\int_{\Omega_m} \eta_j(x)\hat{R}(x,x^i)\,\mathrm{d}\Omega(x) \\ &+ \sum_{h=1}^{H}\int_{\partial\Omega_h} b(x)\left(\frac{\partial \eta_j(x)}{\partial \nu(x)}\right)\hat{P}(x,x^i)\,\mathrm{d}\Gamma(x) \qquad (9)\\ \hat{Q}_i &= \int_{\partial\Omega} \overline{u}(x)T_x\hat{P}(x,y)\,\mathrm{d}\Gamma(x) = \sum_{h=1}^{H}\int_{\partial\Omega_h} \overline{u}(x)T_x\hat{P}(x,x^i)\,\mathrm{d}\Gamma(x), \qquad (10)\\ \hat{D}_i &= \int_{\Omega} g(x)\hat{P}(x,y)\,\mathrm{d}\Omega(x) = \sum_{m=1}^{M}\int_{\Omega_m} g(x)\hat{P}(x,x^i)\,\mathrm{d}\Omega(x). \qquad (11)\end{aligned}$$

Here $\eta_j(x)$ is the shape function.

By defining

$$F(x^i) = \left(1 - d(x^i)\right)\overline{u}(x^i) + \hat{Q}_i + \hat{D}_i, \qquad (12)$$

enables us to indicate (8) as

$$u(x^i) + \sum_{x^j\in\overline{\Omega}} \hat{K}_{ij}u(x^j) = F(x^i), \quad x^i \in \overline{\Omega}. \qquad (13)$$

By setting the following notations

$$I = \delta_{ij}, \quad u = u(x^j), \quad K = -K_{ij}, \quad F = F(x^i),$$

we can then write (13) as equation (14).

$$(I - K)\,u = F. \qquad (14)$$

The system of equations in (14) may be solved iteratively by using Neumann series as given in equation (15) below.

$$u = \sum_{n=0}^{N} K^n F. \qquad (15)$$

By letting

$$f_0 = \mathrm{F}, \quad f_n = K f_{n-1},$$

we can then write (15) as in (16) below.

$$u = u = \sum_{n=0}^{N} K^n \mathrm{F} = \mathrm{F} + \sum_{n=1}^{N} f_n. \tag{16}$$

3. Numerical Experiments

We shall look at three test domains. The first one is a square with vertices' coordinates $(1,1)$, $(2,1)$, $(2,2)$ and $(1,2)$. The second one is a circle with radius 1 and centered at $(2,2)$. The last one is a parallelogram with vertices' coordinates $(3,1)$, $(4,1)$, $(6,2)$ and $(5,2)$. We deliberate in the interior Dirichlet problems as the following cases:

1. $b(x) = 1$.
2. $b(x) = x_2^2$.
3. $b(x) = x_2^4$.
4. $b(x) = x_2^6$.
5. $b(x) = x_2^8$.
6. $b(x) = x_2^{10}$.

For all the cases, we have, $\overline{u}(x) = x_1$ for $x \in \partial\Omega$ and $g(x) = 0$ for $x \in \overline{\Omega}$.

Our work is to conduct an inquiry to check whether an iterative method converges on the Dirichlet BVPs for cases 1-6. Note that the Dirichlet BVPs in case 1 is for PDE with constant coefficient $b(x) = 1$ whereas the Dirichlet BVPS in cases 2-6 are for PDE with variable coefficient $b(x) = x_2^k$, $k = 2, 4, ..., 10$.

When $b(x) = 1$, the BDIDE will be broke up to the Boundary Integral Equation (BIE) that only consists of boundary integrations. The domain integrations disappear due to the fact that the remainder $\hat{R}(x, y)$ and the function $g(x)$ vanish.

Based on the conclusion written in Mikhlin (1957) and Goursat (1964), we can infer that the eigenvalues of Dirichlet BIE operator for constant coefficient are real and stay within an open unit disk. In this paper, we will be progressing to check whether or not its spectral properties prevails for PDE with a few variable coefficients.

The greatest absolute value of the eigenvalues is known as spectral radius. Suppose that spectral radius is less than 1. Then, the spectral radius will reflect the required number of iterations of converging iterative method.

Let $\lambda_j,\ j = 1, 2, ...J$, indicates the list of eigenvalues for the matrix operator K for such

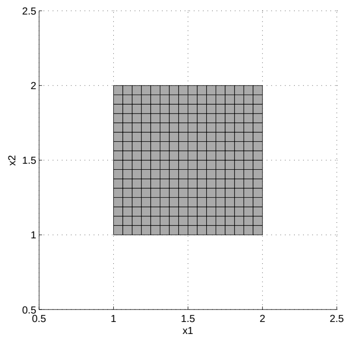
$$(\lambda_j I - K)\, u = 0$$

posses nontrivial solution.

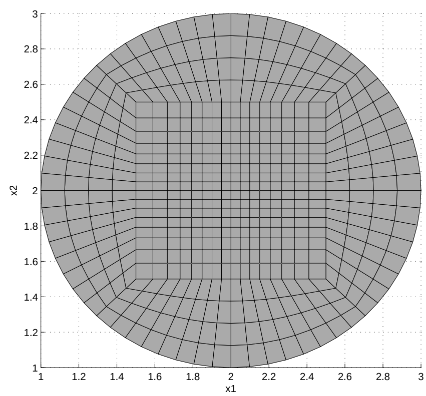
For numerical experiments, we apply Fortran Programming to obtain the operator K in (14). Subsequently, we use MATLAB to get the set of the eigenvalues λ_j for the operator K. In our numerical experiment, we mesh the domains such that the number of nodes for both square and parallelogram are $J = 289$ whereas $J = 545$ for a circle. The illustration of the three test domains that are used in the numerical experiments is shown in Figure 1. From Figures 2-4, it can be noticed that as the number k increases, the modulus of the greatest λ exceeds 1. Figures 2-4 show that the real part of the greatest λ stays within a unit circle but the imaginary part of the greatest λ disperse widely as k increases.

As for the consequences, after some higher k, the Neumann series shall not be reliable to be used on the respective system of equations. Nevertheless, the direct methods e.g. Gaussian elimination can still be employed to solve the system of equations. We note that a few eigenvalues seem to be outside a unit circle after some k, that can result in divergence of the iterative method. Observe that there are repeated eigenvalues visible in the figures. From our computation, it is proven that the eigenvalues are linearly independent.

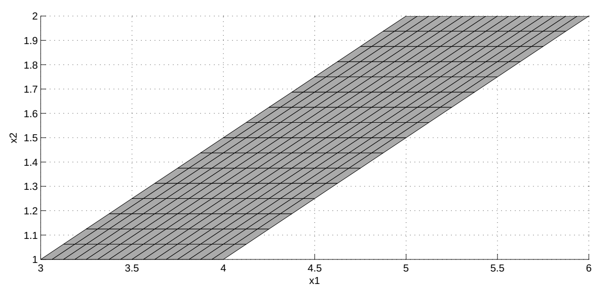
Just like for the coefficient $b\,(x) = 1$, the variable coefficient $b\,(x) = x_k^2,\ \ k = 0,\ 2, \cdots,\ 10$, contribute the real values of eigenvalues that stay between 0 and 1. However, the imaginary values of the eigen- values disperse so much such that $|\mathrm{Im}\lambda_k| < 2$. We study and analyse the correlation between the characteristic size of test domains L, variable coefficient $b\,(x)$ and the spectral radius. Based on the analysis, we conclude that $\max\left|\dfrac{L\nabla b}{b}\right| < 5$ for the greatest eigenvalues to be in a unit circle. Here L be the characteristic size of the test domain.



(a) Square.

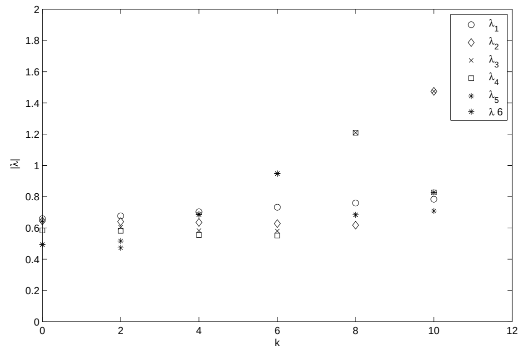


(b) Circle.

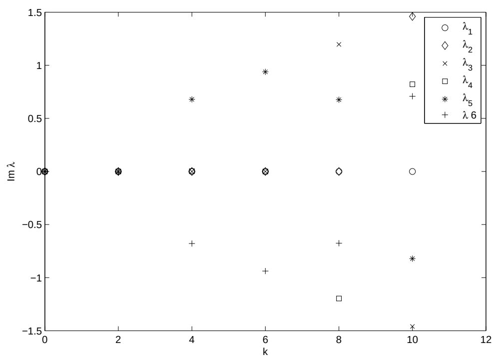


(c) Parallelogram.

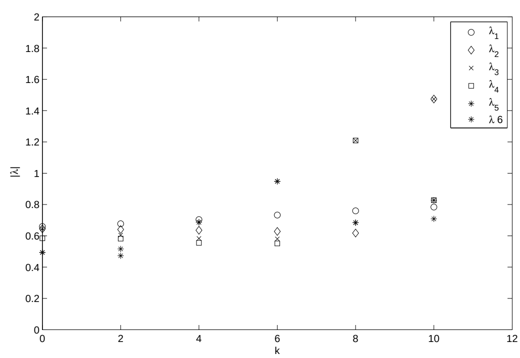
Figure 1: The test domains.



(a) Real λ.

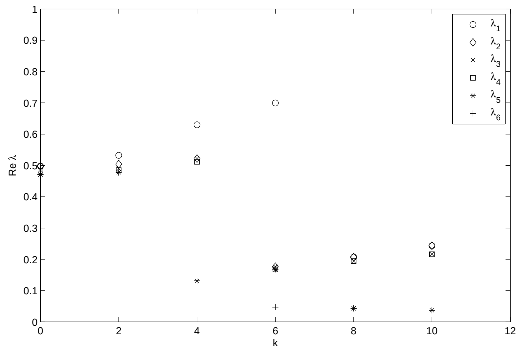


(b) Imaginary λ.

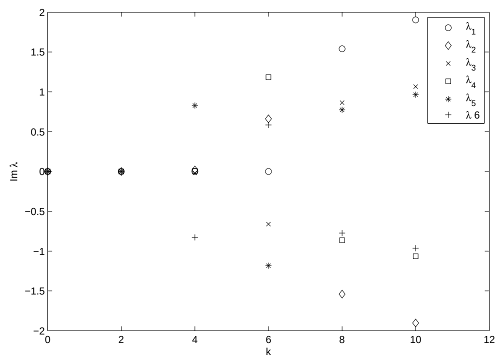


(c) $|\lambda|$.

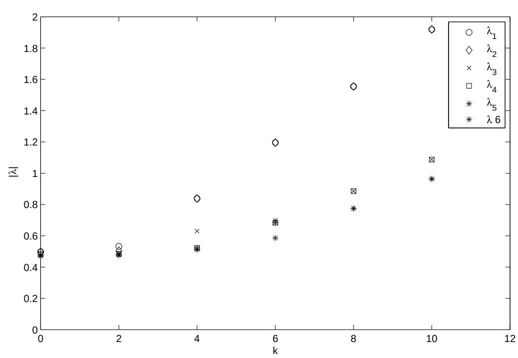
Figure 2: The six greatest eigenvalues of the Dirichlet BDIDE operator K on square vs the power k of $b(x) = x_2^k$.



(a) Real λ.

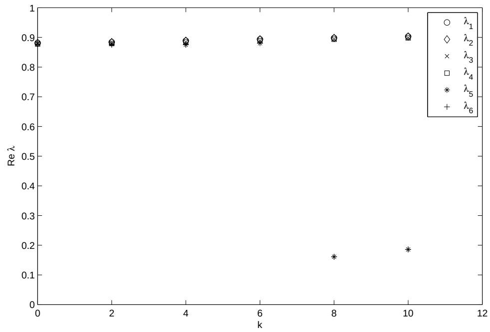


(b) Imaginary λ.

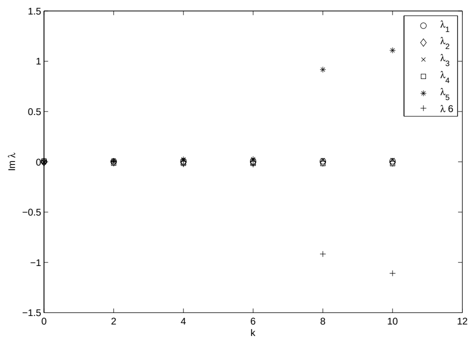


(c) $|\lambda|$.

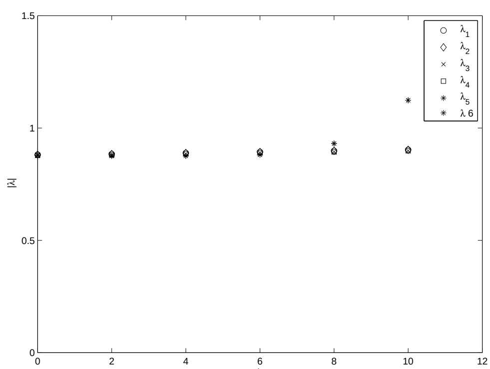
Figure 3: The six greatest eigenvalues of the Dirichlet BDIDE operator K on circle vs the power k of $b(x) = x_2^k$.



(a) Real λ.



(b) Imaginary λ.



(c) $|\lambda|$.

Figure 4: The six greatest eigenvalues of the Dirichlet BDIDE operator K on parallelogram vs the power k of $b(x) = x_2^k$.

4. Conclusions

In this paper, we have observed that the imaginary values of the greatest λ disperse widely whenever the power k of $b(x)$ upsurges to a bigger value. The eigenvalues of the Dirichlet BDIDE's operator disperse widely such that some eigenvalues seem to be outside a unit circle, after some values of k. But, the real values of the greatest eigenvalues mostly sustain and nevertheless appear within the interval $(0, 1)$. For the coefficient $b(x)$ with that big k, the iterative method diverges. However, the direct methods in solving matrix system nevertheless dependable.

Based on the analysis of the results, we reason out that the distribution of the eigenvalues not only rely on the variable coefficient $b(x)$, but also depends on the characteristic size of the domains.

We can estimate the value of the coefficient $b(x)$ relating with the characteristic size L of the domain that guarantees the convergence of the Neumann series in solving the discrete Dirichlet BDIDE.

By having a formula for estimation, it will help the researchers and scientists in choosing the right matrix solver for solving systems of Dirichlet BDIDE based on the characteristic size L of the domain and variable coefficient $b(x)$. This is due to the fact that iterative series will diverges whenever any of the eigenvalues appear to be exterior to a unit circle.

Some conclusions regarding distributions of the eigenvalues for Neumann BDIE operator has been made in Mikhalov and Mohamed (2012). As the Neumann BDIE in Mikhalov and Mohamed (2012), in this paper, we show that the Dirichlet BDIDE also exhibits the spectrum behavior that all the eigenvalues stays within a unit circle whenever $\max\left|\frac{L\nabla b}{b}\right| < 5$. Therefore, from the analysis, there is no much difference between the spectral behaviour of the Neumann BDIE and the Dirichlet BDIDE.

From our analysis for all the test domains which are taken into consideration, whenever $\max\left|\frac{L\nabla b}{b}\right| \geq 5$, a few eigenvalues seem to be exterior to a unit circle. That behavior cause divergence of an iterative method.

It is an advantage that the eigenvalues that we obtained in this experiment are generally complex numbers and their real values constantly less than 1 even for large value of k.

This behavior could lead to a whole lot simpler effort on mapping the outside λ-domain to the outside of the unit circle, with a purpose to result in an amendment of the iterative method as recommended in Kublanovskaya (1959) and Kantorovich and Krylov (1964). The mapping method recommended in Kublanovskaya (1959) and Kantorovich and Krylov (1964) enables the use of the iterative method even when the eigenvalues of the BDIDE operator appear to be exterior to a unit circle.

Therefore, even though the set of the eigen-values that we have obtained in this paper are complex numbers as big value of k, there is still a possibility to use an iterative method in solving the system of equations. As known, any direct method can be used for solving the obtained system of equations. It is interesting if we can compare the efficiency of the direct method and the fast convergence iterative method as in (16).

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