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The System of Equations for Mixed BVP with Three Dirichlet Boundary Conditions and One Neumann Boundary Condition

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Abstract

In this paper, a system of equation for mixed Boundary Value Problem (BVP) on a square shape domain is proposed. We apply Boundary Element Method (BEM) to form a system of equations of two dimensional potential problem which satisfies Laplace equation. The mixed BVP will be reduced to Boundary Integral Equation (BIE) by using direct method which involve Green second identity representation formula. Boundary of the square shape domain is discretized into four linear boundary elements. Each of element is prescribed with specific BC value where Neumann BC is prescribed on one element of the boundary and Dirichlet BC on the other three elements of the boundary. Then, linear interpolation is used on the discretized element. However, there can be two values of Neumann BC at a corner node due to discontinuous outward normal. While, there is only one unknown value for Dirichlet BC at each node. Therefore, our first results for this considered domain yields to the underdetermined system. Hence, we apply formulation presented by Paris and Canas (1997) which applied by them on Dirichlet BVP only. Therefore, we extend the proposed method to handle the discontinuous flux problem for mixed BVP. Finally, we end up with a system of equations.

Keywords: Boundary Element Method, Mixed Boundary Value Problem

Introduction

A BIE of BVP for Partial Differential Equation (PDE) with constant coefficients can be solved by using numerical method. Analytical method can be used to solve only for simple problem related to PDE. Therefore, the numerical method is a good option to approximate solution as mentioned in [4]. The examples of numerical methods used to solve BVPs problems are Finite Difference Method, Finite Element Method (FEM), Finite Volume Method, and Boundary Element Method (BEM). The most common method use is FEM. However, among these methods, as described in [6] and [3], BEM shows some advantages such as the data preparation for BEM are simple. In some cases of BVPs such as those involving infinite domains or 3-dimensional complicated domains, the BEM is more preferable method in solving BVP as compared to FEM. One of the advantage is the BEM reduces the problem dimensionality by one which the discretization of two dimensional (2D) problem only involves the boundary of domain and the discretization for three dimensional (3D) problem only involves the domain of geometry. Unlike FEM, that requires the discretization of the entire domain in 2D problem. As this advantage, it will reduce the computational time in solving the problem. As compared to FEM, the BEM is less time consuming process, but this method also involves some disadvantages such as difficulty in handling singular integrands and we need to know a suitable fundamental solution for each problem.

In this paper, we will concentrate on the mixed BVP for 2D Laplace equation. Green's identity representation formula is used in deriving the reduction of BVPs for PDE with constant coefficients to BIE. We consider a mixed BVP on the square shape domain. Boundary of the domain is divided into four elements where Neumann BC is prescribed on one element of the boundary and Dirichlet BC is prescribed on the other three elements. The aim of this study is to derive the system of equations relating to the considered domain with mixed BC.

Representation Equation for Mixed BVP

Consider a Laplace equation such that u as a potential function which defined in the domain Ω bounded by a boundary $\Gamma := \partial\Omega$. We can write the Laplace equation as

$$Lu(x) := \nabla^2 u(x) = 0, \qquad x \in \Omega. \tag{1}$$

This Laplace equation having mixed BC which is a combination of Dirichlet BC and Neumann BC. Here, Dirichlet BC can be expressed as $u(x) = \bar{u}(x)$ on $\partial\Omega_D$, Neumann BC can be expressed as $t(x) = \dfrac{\partial u(x)}{\partial \boldsymbol{\nu}} = \bar{t}(x)$ on $\partial\Omega_N$, where $\boldsymbol{\nu}$ as an outward normal vector , $\partial\Omega_D$ is a boundary with Dirichlet BC and $\partial\Omega_N$ is a boundary with Neumann BC. Next, we can reduce the Laplace equation in (1) to

BIE through the direct way by using Green theorem as mentioned in [3]. By applying Green theorem, we can reduce the mixed BVP to equation (2) as follows.

$$u(y)=\int_{\Gamma}\left(t(x)U(x,y)-u(x)T(x,y)\right)\,d\Gamma\,. \qquad (2)$$

The notation Γ in equation (2) is denoted as boundary of the domain. The kernels of BIE in (2) i.e. $U(x,y)$ and $T(x,y)$ are defined as

$U(x,y)=-\frac{1}{2\pi}\ln|x-y|=-\frac{1}{2\pi}\ln\, r$ and

$T(x,y)=\frac{\partial U(x,y)}{\partial \boldsymbol{\nu}}=\nu_{x_1}\frac{\partial U(x,y)}{\partial x_1}+\nu_{x_2}\frac{\partial U(x,y)}{\partial x_2}$ for $x,y\in\mathbb{R}^2$. Here, r represents Euclidean distance between load points $y=(y_1,y_2)$ and field points $x=(x_1,x_2)$. The notation $\nu=(\nu_{x_1},\nu_{x_2})$ are outward normal for x_1 and x_2. However, the equation (2) only exist when load point y approaches field points x without considering the situation of singularity such that point y coincides with points x. Therefore, limiting process is carried out for the situation when points y and x coincide. As described in [9], equation (2) comes out with a new term $c(y)$ and we can obtain the BIE as given below.

$$c(y)u(y)=\int_{\Gamma}t(x)U(x,y)d\Gamma(x)-\int_{\Gamma}u(x)T(x,y)\,d\Gamma(x). \qquad (3)$$

The value of $c(y)$ is associated on the position of load point y, see e.g. [7]. Here, point y is located on the boundary. Therefore, $c(y)=\frac{\alpha}{2\pi}$ and α is an interior angle at a corner node y of the boundary, $\partial\Omega$. In this paper, we consider a square shape domain. Therefore, the value of α at a corner point of the boundary is 90^0, then we have $c(y)=\frac{1}{4}$ as mention in [9].

Discretization of BIE for mixed BVP

For mixed BVP, by inserting the prescribed Neumann BC and Dirichlet BC, the BIE (3) can be rearranged where integral with unknown BC value are placed to the Left Hand Side (LHS) of the equation (3), whereas the integral with known prescribed BC values are placed to the Right Hand Side (RHS) of the equation (3). Therefore, the rearrangement of BIE (3) for mixed BVP can be rewritten as follows

$$c(y)u(y)+\int_{\Gamma_N} u(x)T(x,y)\,d\Gamma(x)-\int_{\Gamma_D} t(x)U(x,y)\,d\Gamma(x)= \int_{\Gamma_N} \bar{t}(x)U(x,y)\,d\Gamma(x)-\int_{\Gamma_D} \bar{u}(x)T(x,y)\,d\Gamma(x). \tag{4}$$

Where, Γ_D represents of boundary with Dirichlet BC and Γ_N represents of boundary with Neumann BC. Next, the boundary of the domain needs to be discretized into E linear segments where E equal to 4 elements such that $\partial\Omega \simeq \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$ and nodes are located at the ends of linear elements. Then, we interpolate in each element such that $u(x)=\sum_{j=1}^{J}\phi_j(x)\,u(x^j)$ for $x, x^j \in \partial\Omega$ where $\phi_j(x)$ represents the global shape function, x^j represents the global nodes and J is the number of nodes. By applying the interpolation equation to equation (4) and performing the collocation point x^i at all global nodes, we get the system of equations as follows.

$$c(x^i)u(x^i)+\sum_{x^j\in\Gamma_N} K_{ij}u(x^j)-\sum_{x^j\in\Gamma_D} G_{ij}t(x^j)=\sum_{x^j\in\Gamma_N} \overline{G_{ij}}\;\bar{t}(x^j)-\sum_{x^j\in\Gamma_D} \overline{K_{ij}}\;\bar{u}(x^j)\,,\, x^i\in\partial\Omega. \tag{5}$$

Here, the integrals K_{ij} , G_{ij} , $\overline{K_{ij}}$ and $\overline{G_{ij}}$ can be written as:

$$K_{ij}=\int_{\Gamma_N}\phi_j(x)T(x^i,x)d\Gamma(x)\ , \tag{6}$$

$$G_{ij}=\int_{\Gamma_D}\phi_j(x)U(x^i,x)d\Gamma(x)\ , \tag{7}$$

$$\overline{K_{ij}}=\int_{\Gamma_D}\phi_j(x)T(x^i,x)d\Gamma(x)\ , \tag{8}$$

$$\overline{G_{ij}}=\int_{\Gamma_N}\phi_j(x)U(x^i,x)d\Gamma(x). \tag{9}$$

We define $\mathcal{E}$ to be the set consisting of element number where $\mathcal{E}=\{e\in\mathcal{E}\}=\{1,2,...,E\}$, $\mathcal{D}$ be a set of element numbers with known Dirichlet boundary condition, $\mathcal{D}=\{e\in\mathcal{E}\,|\,e \text{ is an element with prescribed Dirichlet BC}\}$ and $\mathcal{N}$ as a set of element numbers with known Neumann BC, $\mathcal{N}=\{e\in\mathcal{E}\,|\,e \text{ is an element with prescribed Neumann BC}\}$. Here, $\mathcal{D}$ and $\mathcal{N}$ are subsets of $\mathcal{E}$ i.e. $\mathcal{D}\subseteq\mathcal{E}$, $\mathcal{N}\subseteq\mathcal{E}$ and $\mathcal{D}\cap\mathcal{N}=\phi$. We can then discretize the integrals in (6)-(9) so that can be written as

$$K_{ij} = \sum_{e \in \mathscr{N}} \int_{\Gamma_{\mathscr{N}(e)}} \phi_j(x) T(x^i, x) d\Gamma(x), \tag{10}$$

$$G_{ij} = \sum_{e \in \mathscr{D}} \int_{\Gamma_{\mathscr{D}(e)}} \phi_j(x) U(x^i, x) d\Gamma(x), \tag{11}$$

$$\overline{K_{ij}} = \sum_{e \in \mathscr{D}} \int_{\Gamma_{\mathscr{D}(e)}} \phi_j(x) T(x^i, x) d\Gamma(x), \tag{12}$$

$$\overline{G_{ij}} = \sum_{e \in \mathscr{N}} \int_{\Gamma_{\mathscr{N}(e)}} \phi_j(x) U(x^i, x) d\Gamma(x). \tag{13}$$

It is convenient to define the equation (10)-(13) to the local node. In order to do, so we define a relationship between local node and global node i.e. $x^j = x^{j(e,n)} = x_n^e$, for $x_n^e \in \partial\Omega_e$, which implies $u(x^j) = u(x_n^e)$. Then, the point on element can be written in Cartesian coordinates i.e $\begin{Bmatrix} x_1(\eta) \\ x_2(\eta \end{Bmatrix} = \sum_{n=1}^{2} \psi_n(\eta) \begin{Bmatrix} x_{1n}^e \\ x_{2n}^e \end{Bmatrix}$ for $-1 \le \eta \le 1$, with the intrinsic coordinate η as described in [5] and [6]. The term $\psi_n(\eta)$ are represents as linear shape functions for $n = 1$ and $n = 2$ which given respectively, as $\psi_1(\eta) = \frac{1}{2}(1 - \eta)$ and $\psi_2(\eta) = \frac{1}{2}(1 + \eta)$ for $-1 \le \eta \le 1$. Therefore, we can then write the integrals (10)-(13) as $K_{ij} = \sum_{x^j \in \Gamma_{\mathscr{N}(e)}} A_{ni}^e$, $G_{ij} = \sum_{x^j \in \Gamma_{\mathscr{D}(e)}} F_{ni}^e$, $\overline{K_{ij}} = \sum_{x^j \in \Gamma_{\mathscr{D}(e)}} A_{ni}^e$, and $\overline{G_{ij}} = \sum_{x^j \in \Gamma_{\mathscr{N}(e)}} F_{ni}^e$. Here, the integrals A_{ni}^e and F_{ni}^e can be defined as $A_{ni}^e = \int_{-1}^{1} \psi_n(\eta)\, T(x^i, \eta) J_{e1}(\eta) d\eta$ and $F_{ni}^e = \int_{-1}^{1} \psi_n(\eta)\, U(x^i, \eta) J_{e1}(\eta) d\eta$ where n represents local numbering of the nodes on element e i.e. $n = 1, 2$ and J_{e1} in equations A_{ni}^e and F_{ni}^e defines the Jacobian where $J_{e1} = \frac{|\partial\Omega_e|}{2}$ and $|\partial\Omega_e|$ is the length of each boundary element. See e.g. [6]. Here, we consider a square shape domain with mixed BC because it involves corner nodes. We are not interested for the domain bounded with a smooth boundary e.g. a circle domain because there is no discontinuous fluxes at each node. This considered domain is illustrated in figure 1.

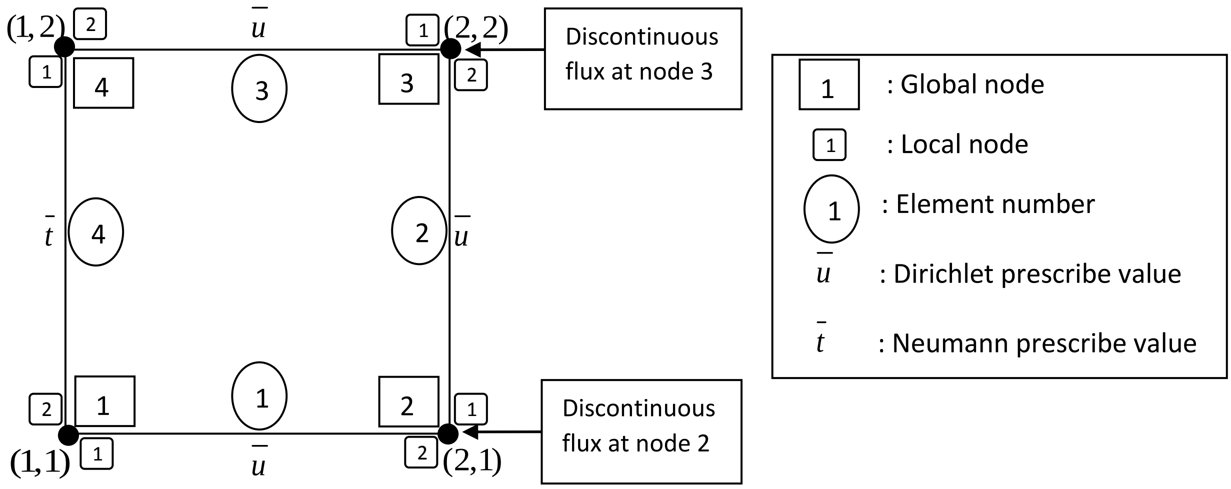

Figure 1. The considered domain with mixed BC with one Neumann prescribe value and three Dirichlet prescribe values.

By referring to the figure 1, $\bar{t}$ is prescribed on fourth element of the boundary $\partial\Omega$ and $\bar{u}$ on the other three elements. Here are the classification for integrals of known values and unknown values which is shown in table 1.

TABLE 1. Integrals of known and unknown values in each elements and nodes for considered domain

	Element 1		Element 2		Element 3		Element 4	
Local nodes	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$	$n=1$	$n=2$
Known values	$\bar{u}_1^1$	$\bar{u}_2^1$	$\bar{u}_1^2$	$\bar{u}_2^2$	$\bar{u}_1^3$	$\bar{u}_2^3$	$\bar{t}_1^4$	$\bar{t}_2^4$
Unknown values	t_1^1	t_2^1	t_1^2	t_2^2	t_1^3	t_2^3	u_1^4	u_2^4

We know that $u_1^1 = u_2^4 = u_1$, $u_2^1 = u_1^2 = u_2$, $u_2^2 = u_1^3 = u_3$ and $u_2^3 = u_1^4 = u_4$. Therefore, the rearrangement in equation (5) yields the following equation

$$c(x_i)u(x_i) - F_{1i}^1 t_1^1 - F_{2i}^1 t_2^1 - F_{1i}^2 t_1^2 - F_{2i}^2 t_2^2 - F_{1i}^3 t_1^3 - F_{2i}^3 t_2^3 = -\left[A_{2i}^4 + A_{1i}^1\right]u_1 - \left[A_{2i}^1 + A_{1i}^2\right]u_2 - \left[A_{2i}^2 + A_{1i}^3\right]u_3 - \left[A_{1i}^4 + A_{2i}^3\right]u_4 + F_{1i}^4 \bar{t}_1^4 + F_{2i}^4 \bar{t}_2^4. \quad (14)$$

Then, the system of equations (14) can be written in the matrix form $B\boldsymbol{x} = H$. The matrix rows and columns of matrix B are representation of collocation points and global nodes, respectively. Vector $\boldsymbol{x}$ represent unknown variables and H represent the known values. Therefore, equation (14) yields the matrix below

$$\begin{bmatrix} -F_{11}^1 & -F_{21}^1 & -F_{11}^2 & -F_{21}^2 & -F_{11}^3 & -F_{21}^3 \\ -F_{12}^1 & -F_{22}^1 & -F_{12}^2 & -F_{22}^2 & -F_{12}^3 & -F_{22}^3 \\ -F_{13}^1 & -F_{23}^1 & -F_{13}^2 & -F_{23}^2 & -F_{13}^3 & -F_{23}^3 \\ -F_{14}^1 & -F_{24}^1 & -F_{14}^2 & -F_{24}^2 & -F_{14}^3 & -F_{24}^3 \end{bmatrix} \begin{bmatrix} t_1^1 \\ t_2^1 \\ t_1^2 \\ t_2^2 \\ t_1^3 \\ t_2^3 \end{bmatrix} = H \,, \qquad (15)$$

where H is the RHS 4×1 matrix, i.e $H = \left[H_i, i = 1,2,3,4\right]$ and i denoted as collocations number, where

$$\begin{aligned} H_i = &-\left[A_{2i}^4 + A_{1i}^1 + c_i\right]u_1 - \left[A_{2i}^1 + A_{1i}^2 + c_i\right]u_2 - \left[A_{2i}^2 + A_{1i}^3 + c_i\right]u_3 \\ &-\left[A_{1i}^4 + A_{2i}^3 + c_i\right]u_4 + F_{1i}^4 \bar{t}_1^4 + F_{2i}^4 \bar{t}_2^4. \end{aligned} \qquad (16)$$

This problem yields an underdetermined system as shown in matrix equation (15). The underdetermined system is a system of linear equations where the number of unknown variables are larger than constraints. Here, underdetermined system is obtained because it involve two discontinuous fluxes problems which is shown in figure 1. A discontinuous flux at the corner node is a situation when there are two unknown fluxes remain associated to a node. See e.g. [9]. In order to handle this problem, we apply formulation presented by Paris and Canas (1997) in [9]. They tried this formula method only on Dirichlet BVP as in Paris and Canas (1997). Therefore, it motivates us to apply the said method on the mixed BVP which will be discussed next. We define an example of a corner node as shown in figure 2.

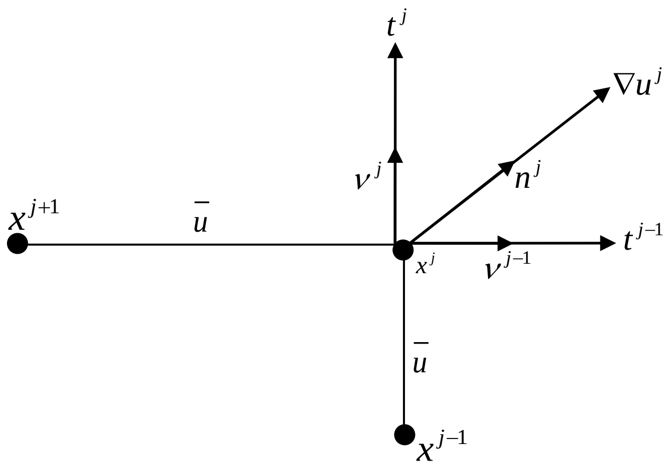

Figure 2. Fluxes at a corner node as described in [9].

From figure 2, prescribed values for Dirichlet BC are known at nodes $j-1$, j and $j+1$. Where j is represented as global node number. Here, we have three coordinates of three distinct nodes i.e. $x^{j-1} = \left(x_1^{j-1}, x_2^{j-1}\right)$, $x^j = \left(x_1^j, x_2^j\right)$ and $x^{j+1} = \left(x_1^{j+1}, x_2^{j+1}\right)$. As described in [9], the equation for two fluxes at j node can be expressed as

$$t_2^{j-1} = \left|\nabla u^j\right| \boldsymbol{n}^j \boldsymbol{\nu}_2^{j-1} \text{ and } t_1^j = \left|\nabla u^j\right| \boldsymbol{n}^j \boldsymbol{\nu}_1^j, \tag{17}$$

where $\boldsymbol{n}^j$ is a unit vector along the gradient direction at node j, $\boldsymbol{\nu}$ is an outward normal vector, $\left|\nabla u^j\right|$ is modulus of the gradient for potential function $\bar{u}$ at node j. It is convenient to denote $D_2^{j-1} = n^j \boldsymbol{\nu}_2^{j-1}$ and $D_1^j = n^j \boldsymbol{\nu}_1^j$ i.e equations (17) can be written as follows

$$t_2^{j-1} = \left|\nabla u^j\right| D_2^{j-1} \text{ and } t_1^j = \left|\nabla u^j\right| D_1^j. \tag{18}$$

In this paper, we have 2 discontinuous fluxes problems. The corner nodes involved are placed at node 2 and 3. Then, by applying equations (18) at node $j = 2$, we obtain

$$t_2^1 = \left|\nabla u^2\right| D_2^1 \text{ and } t_1^2 = \left|\nabla u^2\right| D_1^2. \tag{19}$$

Flux equation for corner node at node $j = 3$, we have

$$t_2^2 = \left|\nabla u^3\right| D_2^2 \text{ and } t_1^3 = \left|\nabla u^3\right| D_1^3. \tag{20}$$

By substituting the new flux equations (19)-(20) in equation (14), we obtain

$$-F_{1i}^1 t_1^1 - \left[F_{2i}^1 D_2^1 + F_{1i}^2 D_1^2\right]\left|\nabla u^2\right| - \left[F_{2i}^2 D_2^2 + F_{1i}^3 D_1^3\right]\left|\nabla u^3\right| - F_{2i}^3 t_2^3 = -\left[A_{2i}^4 + A_{1i}^1 + c_i\right] u_1$$
$$-\left[A_{2i}^1 + A_{1i}^2 + c_i\right] u_2 - \left[A_{2i}^2 + A_{1i}^3 + c_i\right] u_3 - \left[A_{1i}^4 + A_{2i}^3 + c_i\right] u_4 + F_{1i}^4 \bar{t}_1^4 + F_{2i}^4 \bar{t}_2^4. \tag{21}$$

Thus, the system of equation (21) can be written in the matrix form i.e.

$$\begin{bmatrix} -F_{11}^1 & -\left[F_{21}^1 D_2^1 + F_{11}^2 D_1^2\right] & -\left[F_{21}^2 D_2^2 + F_{11}^3 D_1^3\right] & -F_{21}^3 \\ -F_{12}^1 & -\left[F_{22}^1 D_2^1 + F_{12}^2 D_1^2\right] & -\left[F_{22}^2 D_2^2 + F_{12}^3 D_1^3\right] & -F_{22}^3 \\ -F_{13}^1 & -\left[F_{23}^1 D_2^1 + F_{13}^2 D_1^2\right] & -\left[F_{23}^2 D_2^2 + F_{13}^3 D_1^3\right] & -F_{23}^3 \\ -F_{14}^1 & -\left[F_{24}^1 D_2^1 + F_{14}^2 D_1^2\right] & -\left[F_{24}^2 D_2^2 + F_{14}^3 D_1^3\right] & -F_{24}^3 \end{bmatrix} \begin{bmatrix} t_1^1 \\ \left|\nabla u^2\right| \\ \left|\nabla u^3\right| \\ t_2^3 \end{bmatrix} = H, \tag{22}$$

where H is the RHS 4×1 matrix, i.e. $H = \left[H_i, i = 1,2,3,4\right]$, where

$$H_i = -\left[A_{2i}^4 + A_{1i}^1 + \frac{1}{4}\right] u_1 - \left[A_{2i}^1 + A_{1i}^2 + \frac{1}{4}\right] u_2 - \left[A_{2i}^2 + A_{1i}^3 + \frac{1}{4}\right] u_3$$
$$-\left[A_{1i}^4 + A_{2i}^3 + \frac{1}{4}\right] u_4 + F_{1i}^4 \bar{t}_1^4 + F_{2i}^4 \bar{t}_2^4.$$

Now, we can solve the matrix equation to obtain the value of unknown variables in the matrix equation (22) i.e t_1^1, $\left|\nabla u^2\right|$, $\left|\nabla u^3\right|$ and t_2^3. This matrix system can then be solved by using direct or iterative methods to get the approximation values for unknown variables. Hence, we used the approximation values to get mixed BVP values.

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Conclusion

As for conclusion, we have done the theoretical development of the system of equations for mixed BVP with three Dirichlet BC and one Neumann BC. In this paper, we only consider a simple mixed BVP on a square domain in figure 1 where one Neumann BC is prescribed on element 4 and Dirichlet BC on the other three elements. This considered mixed BVP led to an underdetermined matrix system due to the discontinuous fluxes problems if we use standard BEM. Therefore, we study the method proposed by Paris and Canas (1997) which was applied on Dirichlet BVP only. In this paper, we extend the method proposed by Paris and Canas (1997) on the mixed BVP and come out a matrix system. However, the behavior of the matrix equation in (28) need to be investigated in order to make sure the system of equations (27) can be solved numerically and not an ill conditioned matrix.

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