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IMPLEMENTATIONS OF BOUNDARY–DOMAIN INTEGRO-DIFFERENTIAL EQUATION FOR DIRICHLET BVP WITH VARIABLE COEFFICIENT

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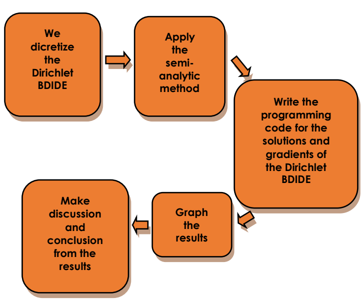
Graphical abstract



Abstract

In this paper, we present the numerical results of the Boundary-Domain Integro-Differential Equation (BDIDE) associated to Dirichlet problem for an elliptic type Partial Differential Equation (PDE) with a variable coefficient. The numerical constructions are based on discretizing the boundary of the problem region by utilizing continuous linear iso-parametric elements while the domain of the problem region is meshed by using iso-parametric quadrilateral bilinear domain elements. We also use a semi-analytic method to handle the integration that exhibits logarithmic singularity instead of using Gauss-Laguare quadrature formula. The numerical results that employed the semi-analytic method give better accuracy as compared to those when we use Gauss-Laguerre quadrature formula. The system of equations that obtained by the discretized BDIDE is solved by an iterative method (Neumann series expansion) as well as a direct method (LU decomposition method). From our numerical experiments on all test domains, the relative errors of the solutions when applying semi-analytic method are smaller than when we use Gauss-Laguerre quadrature formula for the integration with logarithmic singularity. Unlike Dirichlet Boundary Integral Equation (BIE), the spectral properties of the Dirichlet BDIDE is not known. The Neumann iterations will converge to the solution if and only if the spectral radius of matrix operator is less than 1. In our numerical experiment on all the test domains, the Neumann series does converge. It gives some conclusions for the spectral properties of the Dirichlet BDIDE even though more experiments on the general Dirichlet problems need to be carried out.

Keywords: Boundary-domain integro-differential equation, dirichlet problem, partial differential equation, semi-analytic integration method

1.0 INTRODUCTION

It is well known that Boundary Element Method (BEM) can be used to solve Boundary Value Problems (BVPs) describes by PDE with constant coefficient numerically.

At first, a BVP for PDE with constant coefficient need to be transformed to a Boundary Integral Equation (BIE). The transformation is only feasible on the condition that a PDE's fundamental solution is known. This fundamental solution satisfies the PDE with constant coefficient exactly. The numerical computational method for solving the BIE is known as BEM. By using BEM, the dimensionality of the BVPs is reduced by one order. The details on BEM are provided in e.g. 0, 0. However, the BEM is not applicable for problems with fundamental solution is unknown like occurs in problems for PDE with variable coefficient.

A parametrix is much obtainable than a fundamental solution. It has been discussed in [3, 4, 5] that the Neumann or Dirichlet problems for PDE with variable coefficient can be reduced to Boundary-Domain Integral Equation (BDIE) or BDIDEs, respectively. The numerical computational of BDIE or BDIDEs is known as Boundary-Domain Element Method (BDEM).

We take into account the Dirichlet problem for the following second-order elliptic PDE.

$$Au(x)=\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_j}\right]=f(x),\quad x\in\Omega, \tag{1}$$

$$u(x):=\bar{u}(x),\quad x\in\partial\Omega, \tag{2}$$

where $\partial\Omega$ is the boundary, Ω such that $\Omega\subset\mathbb{R}^2$ is bounded domain, $u(x)$ is the unknown function, $a(x)>0$ is the prescribed variable coefficient, while $f(x)$ and $\bar{u}(x)$ are the prescribed functions.

A parametrix for (1) is as follows.

$$P(x,y)=\frac{\ln|x-y|}{2\pi a(y)},\quad x,y\in\mathbb{R}^2, \tag{3}$$

where the radius r is as follows:

$$r=|x-y|=\sqrt{(x_i-y_i)(x_i-y_i)}\ . \tag{4}$$

The parametrix (3) satisfies the following equation:

$$A_xP(x,y)=\delta(x-y)+R(x,y),$$

where

$$R(x,y)=\frac{1}{2\pi a(y)}\sum_{i=1}^{2}\frac{x_i-y_i}{|y-x|}\frac{\partial a(x)}{\partial x_i},\quad x,y\in\mathbb{R}^2, \tag{5}$$

and $\delta(x-y)$ is the Dirac delta function.

We suppose that

$$Tu(x)=\sum_{j=1}^{2}a(x)\nu_j(x)\frac{\partial u(x)}{\partial x_j},$$

$$T_xP(x,y)=\sum_{j=1}^{2}a(x)\nu_j(x)\frac{\partial P(x,y)}{\partial x_j}=\sum_{j=1}^{2}\frac{a(x)\nu_j(x)(x_j-y_j)}{2\pi a(y)r^2}.$$

Here $\nu(x)=(\nu_1(x),\nu_2(x))$ is normal to Ω which is pointing away from the boundary $\partial\Omega$.

As described in 0, 0, the direct united Dirichlet BDIDE is given as follows:

$$\begin{aligned}&c(y)u(y)+\int_{\Omega}R(x,y)u(x)\,\mathrm{d}\Omega(x)+\int_{\partial\Omega}P(x,y)Tu(x)\,\mathrm{d}\Gamma(x)\\&=\int_{\partial\Omega}\bar{u}(x)T_xP(x,y)\,\mathrm{d}\Gamma(x)+\int_{\Omega}P(x,y)f(x)\,\mathrm{d}\Omega(x),\ y\in\bar{\Omega}.\end{aligned} \tag{6}$$

The coefficient $c(y)$ depends on the position of point y i.e.

$$c(y)=\begin{cases}1 & \text{if } y\in\Omega,\\ 0 & \text{if } y\in\mathbb{R}^2\setminus\bar{\Omega},\\ \alpha(y)/2\pi & \text{if } y\in\partial\Omega,\end{cases}$$

where $\bar{\Omega}=\Omega\cup\partial\Omega$ and $\alpha(y)$ is the interior angle at $y\in\partial\Omega$. When $\partial\Omega$ is a smooth boundary, the angle $\alpha(y)=\pi$ that gives $c(y)=1/2$.

Some analysis of direct united BDIDE were given in e.g. 0 where they discussed the existence, uniqueness and invertibility of the BDIDE operator. Whereas, the regularity and asymptotic behavior of the solutions obtained by BDIDE were detailed in 0.

In 0, 0, the numerical implementation of the Neumann BDIE with added perturbation operator for PDE with variable coefficient was presented.

The system of linear equations that obtained from the discretized Neumann BDIE was solved by a direct an iterative methods. In 0, the spectral properties of the discrete Neumann BDIE operator by analyzing the maximal eigen-values was also presented.

In order to minimalize the integration error for the integration with logarithmic kernel e.g. $P(x,y)$, Mohamed in 0 constructed the semi-analytic integration method. It was envisaged that the proposed semi-analytic integration method will make the numerical solution approaches the exact of the respected BDIDE's solution closer than the standard numerical approach involving logarithmic singularity i.e. Gauss-Laguerre quadrature formula. However, no numerical implementation was presented in 0 to validate the claim.

In this paper, we present on the numerical solutions of the Dirichlet BDIE (6). All the numerical procedure is based on the suggestion made in 0.

2.0 DISCRETIZATION OF BDIDE

We integrate the integrals in (6) by using the standard Gauss-Legendre quadrature formula whenever no singularity is involved. Whenever the singularities occur in the integrals, special care must be taken for their evaluations. For singularities anticipate on the domain integrals in (6), we use the Duffy transformation.

The boundary integral on the left hand side of (6) exposes the disadvantage of the logarithmic singularity of $P(x,y)$ whenever $x=y$.

The integral that consists of logarithmic singularity cannot be accurately and efficiently be computed by using the standard Gauss-Legendre quadrature formula. Normally Gauss-Laguerre quadrature formula can treat the logarithmic singularity. See e.g. 0. However, the approach might lead to the use of considerably many quadrature points in order to get an accurate result.

The semi-analytic method proposed in 0 was derived as an alternative to Gauss-Laguerre formula for evaluating integral that involves with logarithmic singularity.

Here, we present the numerical results of Dirichlet direct-united BDIDE to validate the claim made. By following the recommendation made in 0, we mesh the domain Ω by M quadrilateral bilinear elements e_m i.e. $\overline{\Omega} = \bigcup_m^M \overline{e}_m,\ e_k \cap e_m = \varnothing,\ k \neq m.$

Suppose that $\xi =: (\xi_1, \xi_2)$ represents the intrinsic coordinate for square element such that $-1 \le \xi_1, \xi_2 \le 1.$

We can relate the intrinsic coordinate ξ and the Cartesian coordinate $x(\xi)$ on the quadrilateral domain element $e_m \subset \Omega$ as follows.

$$x(\xi) = \sum_{N=1}^{4} \Phi_N(\xi) X^{mN}, \tag{7}$$

where

$$\Phi_1(\xi) = (1-\xi_1)(1-\xi_2)/4, \quad \Phi_2(\xi) = (1+\xi_1)(1-\xi_2)/4,$$

$$\Phi_3(\xi) = (1+\xi_1)(1+\xi_2)/4, \quad \Phi_4(\xi) = (1-\xi_1)(1+\xi_2)/4,$$

and X^{mN} for $N = 1,\ldots,4$ is the vertices of the quadrilateral elements e_m. For the boundary, we discretized the boundary $\partial\Omega$ by using L continuous linear isoparametric elements, i.e. $\partial\Omega = \mathrm{U}_l^L \partial\Omega_l$. Here $\partial\Omega_l$ is the line segment associates with the outer side of quadrilateral domain element e_m.

We denote η be the intrinsic coordinate on the reference segment such that $-1 \le \eta \le 1.$ Then, the relation between the intrinsic coordinate η and the Cartesian coordinate $x(\eta)$ on the boundary element $\partial\Omega_l$ is given as follows:

$$x(\eta) = \sum_{n=1}^{2} \Psi_n(\eta) X^{ln}, \tag{8}$$

where $\Psi_n(\eta)$ are the local one-dimensional shape functions given below.

$$\Psi_1(\eta) = \frac{1}{2}(1-\eta), \quad \Psi_2(\eta) = \frac{1}{2}(1+\eta), \quad -1 \le \eta \le 1.$$

Equation (6) is equivalent to the equation given below.

$$\begin{aligned} u(y) &+ \int_\Omega R(x,y)u(x)\,\mathrm{d}\Omega(x) + \int_{\partial\Omega} P(x,y)Tu(x)\,\mathrm{d}\Gamma(x) \\ &= \left(1-c(y)\right)\overline{u}(y) + \int_{\partial\Omega} \overline{u}(x) T_x P(x,y)\,\mathrm{d}\Gamma(x) \\ &+ \int_\Omega P(x,y) f(x)\,\mathrm{d}\Omega(x), \quad y \in \overline{\Omega}. \end{aligned} \tag{9}$$

Let the solution be sought at J points such that we will have J number of node points $x^j \in \overline{\Omega}$. Applying interpolation to (9), we obtain

$$\begin{aligned} & u(x^i) + \sum_{x^j \in \overline{\Omega}} K_{ij}^D u(x^j) = \left(1-c(x^i)\right)\overline{u}(x^i) + Q_i^D + D_i^D, \\ & x^i \in \overline{\Omega},\ x^j \in \overline{\Omega},\ j = 1,2,\ldots,J, \end{aligned} \tag{10}$$

where

$$K_{ij}^D = \sum_{x^j \in \overline{\Omega}_m} G_{N(j,m),i}^m + \sum_{\{x^j \in \overline{\Omega}_m : \overline{\Omega}_m\} \supset \partial\Omega_l} A_{N(j,m),i}^l,$$

$$Q_i^D = \sum_{l=1}^{L} F_i^l,$$

$$D_i = \sum_{m=1}^{M} F_i^m.$$

Here $n(j,l)$ is the local number associates with the node x^j on $\partial\Omega_l$, $N(j,m)$ is the local number of the node x^j on e_m, and the notations $G_{N,i}^m$, $A_{N,i}^l$, F_i^l and H_i^m are represented by the following integrals:

$$G_{N,i}^m = \int_{-1}^{1}\int_{-1}^{1} \Phi_N(\xi) R(x(\xi), x^i) J_{m2}(\xi)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{11}$$

$$A_{N,i}^l = \int_{-1}^{1} P(x(\eta), x^i) \left[a(x(\eta)) \left(\sum_{p=1}^{2}\sum_{k=1}^{2} \frac{\partial \Phi_N}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_p}\bigg|_{\xi=\xi(\eta)} \nu_p(x(\eta)) \right) \right] J_{l1}(\eta)\,\mathrm{d}\eta. \tag{12}$$

$$F_i^l = \int_{-1}^{1} \overline{u}(x(\eta)) T_x P(x(\eta), x^i) J_{l1}(\eta)\,\mathrm{d}\eta, \tag{13}$$

$$H_i^m = \int_{-1}^{1}\int_{-1}^{1} P(x(\xi), x^i) f(x(\xi)) J_{m2}(\xi)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2. \tag{14}$$

Here J_{m2} and J_{l1} are the Jacobians for the transformation in relations (7) and (8), respectively, as stated in equations (15) and (16) below.

$$J_{l1} = |\partial\Omega_l|/2, \tag{15}$$

$$J_{m2}(\xi) = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1}, \quad 1 \le m \le M. \tag{16}$$

All the integrals in (11)-(14) are calculated by the Gauss Legendre quadrature formulas whenever the singularity does not exhibit i.e. for the collocation point $x^i \neq x^j$. Special care must be taken whenever the singularity occurs i.e. when $x^i = x^j$. The boundary integral in (12) with the kernel involving $P(x(\eta), x^i)$ that expresses logarithmic singularity can be calculated by using Gauss-Laguerre quadrature formula. However, Gauss-Laguerre quadrature formula might not be a good choice since it demands large number of quadrature points for an accurate result. We use the semi-analytic formula proposed in 0 as an alternative in handling the logarithmic singularity in (12).

3.0 NUMERICAL IMPLEMENTATION

The numerical implementation was done by using Fortran. We solve the system of equations (10) by employing both direct and iterative methods.

The iterative method that we consider is a Neumann series expansion such that (10) is arranged as $I - K \;\; u = \mathrm{F}$, where

$$I=\delta_{ij},\quad u=u(x^j),\quad K=-K_{ij},$$

$$\mathrm{F}\ =F(x^j)=\left(1-c(x^i)\right)\bar{u}(x^i)+Q_i^D+D_i^D.$$

The solution u then can be written as Neumann series expansion i.e.

$$u=\sum_{n=0}^{N}K^n\mathrm{F}\ . \tag{17}$$

$$u=u=\sum_{n=0}^{N}K^n\mathrm{F}\ =\mathrm{F}\ +\sum_{n=1}^{N}g_n. \tag{18}$$

We use the same test domains that have been used in 0 i.e. a square $1<x_1,x_2<2$, a unit circle that centered at (2, 2) and a parallelogram with (3, 1), (4, 1), (6, 2) and (5, 2) as the vertices. Two interior Dirichlet boundary value problems with the following parameters are considered.

Test 1: $a(x)=x_2,\ f(x)=0 \text{ for } x\in\Omega\cup\partial\Omega,$
$\bar{u}(x)=x_1 \text{ for } x\in\partial\Omega,$

Test 2: $a(x)=x_2^2,\ f(x)=2x_2^2 \text{ for } x\in\Omega\cup\partial\Omega,$
$\bar{u}(x)=x_1^2 \text{ for } x\in\partial\Omega.$

In order to check the accuracy the solutions and its gradients, we use posteriori relative errors i.e.,

$$\epsilon(u)=\frac{\max\limits_{1\le j\le J}\left|u_{estimation}(x^j)-u_{exact}(x^j)\right|}{\max\limits_{1\le j\le J}\left|u_{exact}(x^j)\right|},$$

$$\epsilon(\nabla u)=\frac{\max\limits_{1\le j\le J}\left|\nabla u_{estimation}(x_c^m)-\nabla u_{exact}(x_c^m)\right|}{\max\limits_{1\le j\le J}\left|\nabla u_{exact}(x_c^m)\right|},$$

where x_c^m be the centers of e_m.

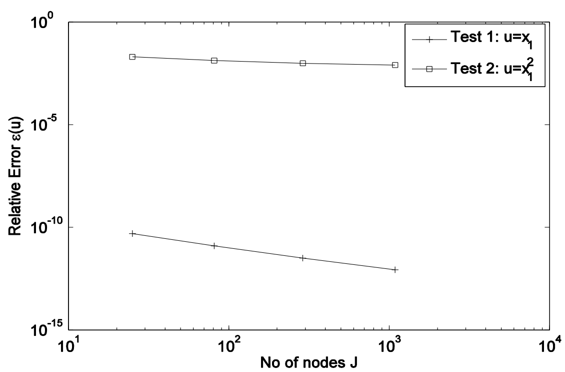


Figure 1 Relative errors of estimated solutions against number of nodes J for square domain

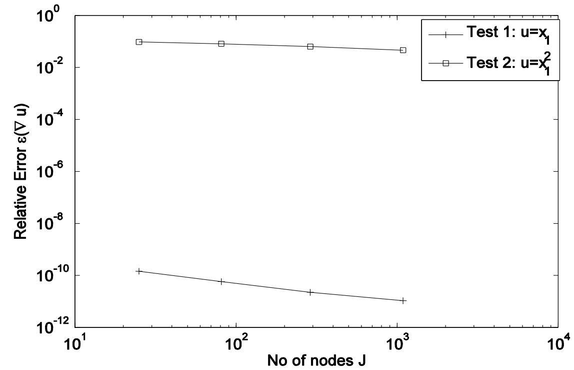


Figure 2 Relative errors of the estimated gradients against number of nodes J for square domain

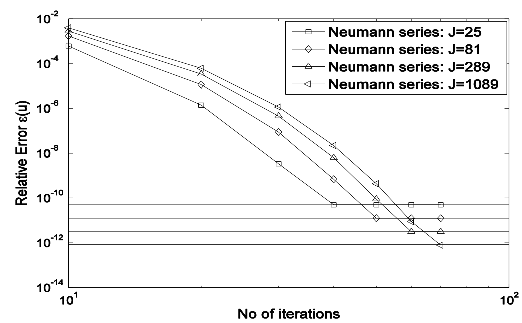


Figure 3 Relative errors of the estimated solutions against number of Neumann iterations for Test 1 comparatively to the one obtained by the direct method (LU decomposition) that represented by the horizontal lines, for various number of nodes J on square

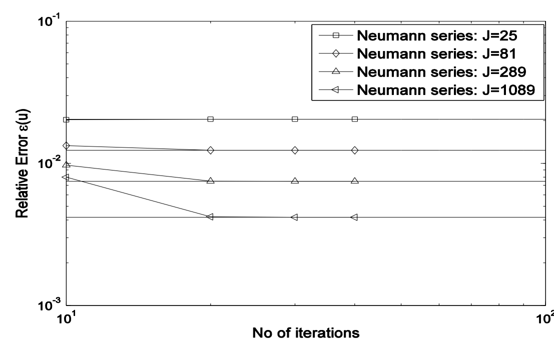


Figure 4 Relative errors of the estimated solutions against number of Neumann iterations for Test 2, comparatively to the one obtained by the direct method (LU decomposition) that represented by the horizontal lines, for various number of nodes J on the square

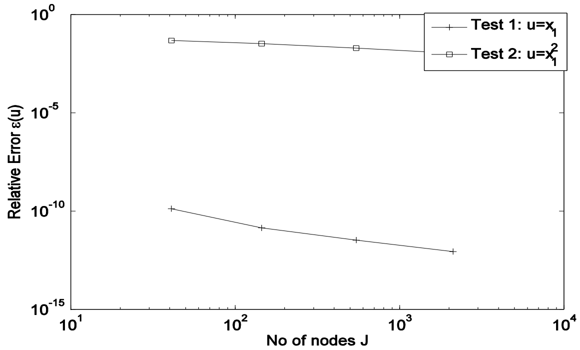


Figure 5 Relative errors of estimated solutions against number of nodes J for circle

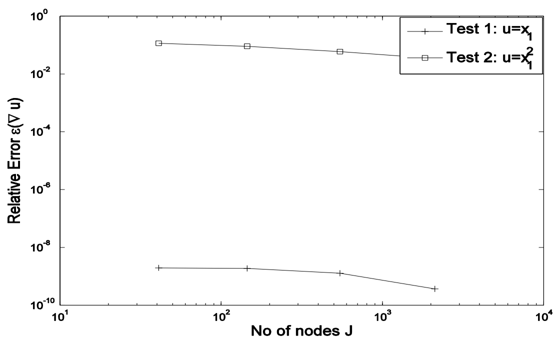


Figure 6 Relative errors of the estimated gradients against number of nodes J for circle

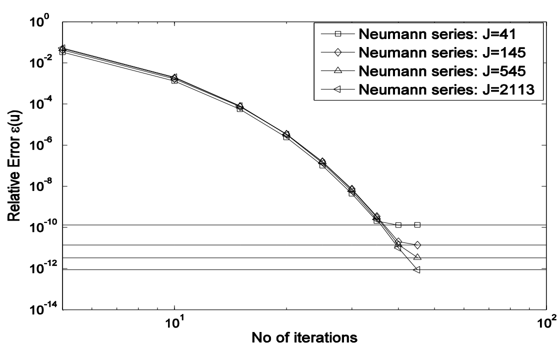


Figure 7 Relative errors of the estimated solutions against number of Neumann iterations for Test 1 comparatively to the one obtained by the direct method (LU decomposition) that represented by the horizontal lines, for various number of nodes J on circle

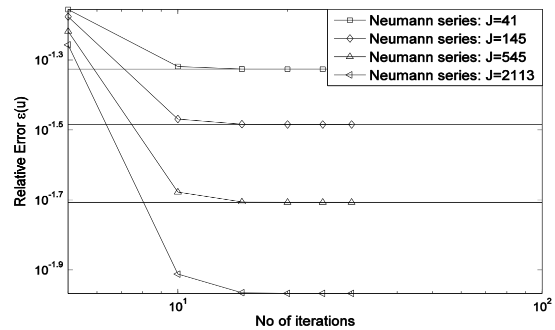


Figure 8 Relative errors of the estimated solutions against number of Neumann iterations for Test 2 comparatively to the one obtained by the direct method (LU decomposition) that represented by the horizontal lines, for various number of nodes J circle

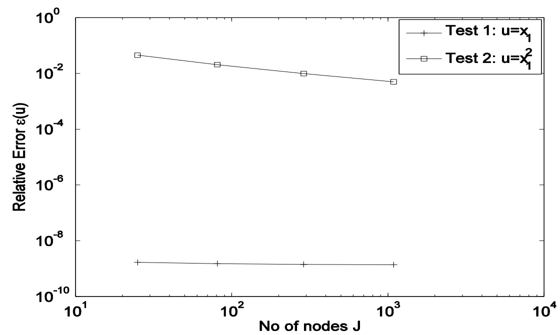


Figure 9 Relative errors of estimated solutions against number of nodes J for parallelogram

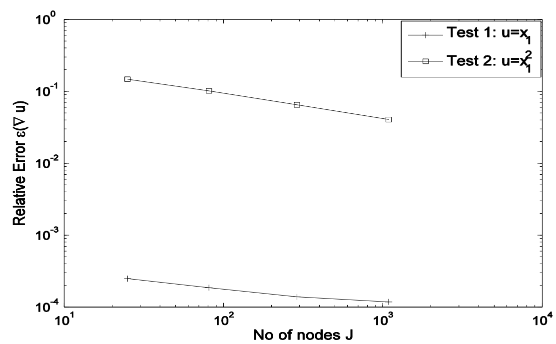


Figure 10 Relative errors of the estimated gradients against number of nodes J for parallelogram

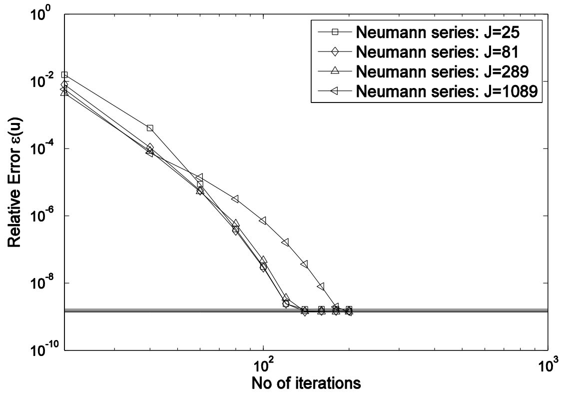


Figure 11 Relative errors of the estimated solutions against number of Neumann iterations for Test 1 comparatively to the one obtained by the direct method (LU decomposition) that represented by the horizontal lines, for various number of nodes J on parallelogram

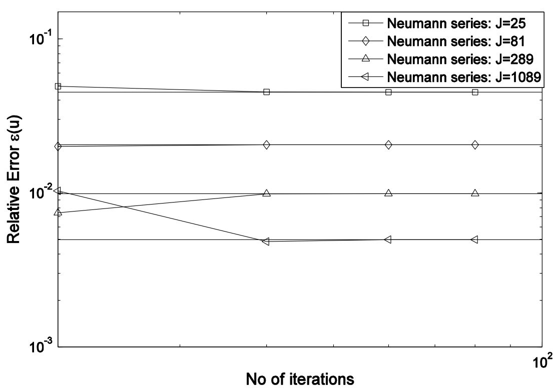


Figure 12 Relative errors of the estimated solutions against number of Neumann iterations for Test 2 comparatively to the one obtained by the direct method (LU decomposition) that represented by the horizontal lines, for various number of nodes J on parallelogram

As the nodes increases, we can see from Figures 1, 2, 5, 6, 9 and 10 that the relative errors $\epsilon(u)$ and $\epsilon(\nabla u)$ decline. Whereas from Figures 3, 4, 7, 8, 11 and 12, we observe that the iterative solutions converge to the direct method's solutions.

Based on Figures 3, 4, 7, 8, 11 and 12, we see that the iterative solutions converges around 20-100 iterations according to test domains and the test problems. Simpler domain like square requires less number of iterations i.e. around 40-70 iterations for Test 1 and 20-30 iterations for Test 2. Less number of iterations is needed for Test 2 since the accuracy of the direct method's solutions as comparison is lower.

For circle, we need about 40-50 iterations for Test 1 and 20-30 iterations for test 2. Whereas for parallelogram, the number of iterations requires is 50-100 iterations for Test 1 and 40-60 iterations for Test 2.

Let $\epsilon(u) \sim J^{-q/2} \sim h^{q}$, $\nabla\epsilon(u) \sim J^{-q'/2} \sim h^{q'}$,

where h is the size of the elements (diameter).

The approximation value of the convergence rate q for $\epsilon(u)$, and the approximation value of the convergence rate q' or $\nabla\epsilon(u)$ associate with each test domain are shown in Table 1 and Table 2, respectively.

Table 1 The value of q

Test domain	Test 1	Test 2
Square	1	1.5
Circle	0.6	1
Parallelogram	1	2

Table 2 The value of q'

Test domain	Test 1	Test 2
Square	0.16	0.9
Circle	0.08	0.4
Parallelogram	0.05	0.9

4.0 CONCLUSION

In this paper, the numerical results for discrete Dirichlet BDIDE operator show that the convergence rates for solution $\varepsilon(u)$ as in Table 1 are close to the discrete Neumann BDIE results as presented in 0 i.e. $q \approx 1$ and $q \approx 2$, respectively for Test 1 and Test 2. Thus, the results produce approximately linear and quadratic convergences rate of solution u with reference to the element size h for Test 1 and Test 2, respectively. Thus in both problems, Dirichlet and Neumann produce almost the same convergence rates of estimated solutions.

We have also validated from the numerical experiments that high accuracy results have been achieved by using the semi-analytic method in order to tackle the singularity cause by integration of logarithmic, as proposed in 0. Therefore, we have deduced that this semi-analytic integration methods is a good alternative to the Gauss-Laguerre quadrature formula for calculating integrations that involve logarithmic singularity.

Even though the convergence of the iterative solutions are confirmed for both test problems, deeper analysis on the spectrum of the BDIDE operator needs to be carried out in order to draw a conclusion on general Dirichlet problems. However, the results give an idea on the maximal eigen-values for the BDIDE operator.

Therefore, it would be interesting if the research on the analysis of the spectral properties for the Dirichlet BDIDE is conducted. This is useful as to validate that that the iterative method can be used to solve discrete Dirichlet BDIDE on various shapes of domains and general problems.

Other than that, it is also suggested that for the next research work to do the comparison for results obtain from using semi-analytic method as suggested in 0 and the Gauss-Laguerre quadrature formula to integrate the integral that involve the singularity of a logarithmic function.

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