The Spectrum Of Discretized Bdides With Reduced Number Of Collocation Points

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Abstract
The behaviour of the eigenvalues of Dirichlet Boundary-Domain Integro-Differential Equations (BDIDEs) with a reduced number of collocation points has never been discussed theoretically. The uncertainty of the behaviour of the eigenvalues of Dirichlet BDIDEs will prohibit the use of iteration methods in solving the BDIDEs system of equations. The purpose of this paper is to demonstrate the spectral properties of matrix operator obtained from the discretized Dirichlet BDIDEs with reduced number of collocation points. We calculate the eigenvalues of the matrix operator, numerically. The discussions of the spectral properties are based on the eigenvalues of the discretized BDIDEs that are obtained numerically. In our numerical test, the attribution of the eigen-values for matrix operator obtained numerically for the discretized BDIDEs with reduced number of collocation points exceeds 1. The findings demonstrate that it is utterly impracticable to solve the system yielded from the discretized Dirichlet BDIDEs with a reduced number of collocation points with an iterative method. The theoretical explanation of why this behaviour occurs is also provided. With this result of the eigenvalues attained, the matrix equations yielded from the discretized BDIDEs with a reduced number of collocation points can purely be solved by direct methods.

Keywords: Direct united boundary-domain integro-differential equation, Dirichlet problem, partial differential equation, semi-analytic integration method.

1. INTRODUCTION

Many real-life physical problems like those involving sound, elasticity, and fluid flow can be expressed as Partial Differential Equations (PDEs). However, almost all the real-life problems are very complicated that the analytical solutions of the problems are almost impossible to attain. There are many numerical methods for PDE are available. The details of the numerical methods in solving PDEs are in, e.g. [1]-[4].

The Boundary Element Method (BEM), one of the numerical method requires representation formulas i.e. Green's identity and Betti's formula, respectively, for potential and elasticity theories. The representation formulas are in boundary integration representation and known as Boundary Integral Equations (BIEs). These representation formulas can be attained by making use of fundamental solutions.

However, the fundamental solutions are only available for Boundary Value Problems (BVPs) related to PDE associates with a constant coefficient. Instead of a fundamental solution, one can use a parametrix to obtain the representation formula for to a problem related to PDE with a variable coefficient. The representation formula is in Boundary-Domain representation formulas, which are known as Boundary-Domain Integral Equations (BDIEs) and BDIDEs, respectively, for Neumann and Dirichlet problems.

The numerical implementation for Neumann BDIE and the details on the attribution of the eigenvalues for matrix operator belongs to the system of equations for discretized Neumann BDIEs is presented in [5]. In [6], the two approaches of Dirichlet BDIDEs in the interpolations process are shown.

For the first approach, we take the collocation points at all nodes. The second approach deals with taking the collocation points for the nodes in the interior domain only. Then, also in 2016, the numerical solution of the Dirichlet BDIDEs by using the first approach in the interpolation process was presented in [7]. In 2019, [8] presented the discussion on the spectrum, i.e., the set of the eigenvalues of the Dirichlet BDIDEs for PDE with variable coefficient by employing the first approach of the interpolation process.

However, the attribution of the eigenvalues for matrix operator belongs to the BDIDEs system of equations with less no collocation points as in [6] has never been discussed until now. The fact that the system has a reduced number of collocation points compared to the system of BDIDEs obtained from the first approach gives an advantage in terms of computational effort. Therefore, the second approach is more interesting for numerical purposes. The numerical results for Dirichlet BDIDEs using the second approach were also shown in [6], which also utilized the semi-analytic method as introduced in [9].

The solution of the system of the Dirichlet BDIDEs can be attained by direct methods e.g. LU decomposition and Gaussian elimination, regardless of the behaviour of the eigenvalues of the system operator. But it is not the case for the application of the iteration methods. The use of iteration methods depends on the eigenvalues' behaviour such that all of them lie in the unit circle. However, the attribution of the eigen-values for matrix operator of its system of equations are not discussed in that paper.

In this paper, we extend the discussion of Dirichlet BDIDE by presenting the calculated eigenvalues of Dirichlet BDIDEs matrix operator with reduced collocation points. The deliberation on the numerical eigenvalues' behaviour corresponds to the theoretical studies is also presented in this paper.

2. RESEARCH METHOD

Let us take into account the linear second-order elliptic PDE as below.

$$Au(x)=\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_j}\right]=f(x),\quad x\in\Omega, \tag{1}$$

with a Dirichlet boundary condition

$$u(x)=\bar{u}(x),\quad x\in\partial\Omega.$$

Here $u(x)$ be the inquired function whereas $f(x)$ and $\bar{u}(x)$ be known functions. Note that equation (1) is a PDE with $a(x)$ be a variable coefficient that prohibits us from using a fundamental solution. Instead, we can use a parametrix

$$P(x,y)=\frac{\ln|x-y|}{2\pi a(y)},\quad x,y\in\mathbb{R}^2, \tag{2}$$

as the substitute for a fundamental solution.

It is stated in, e.g. [6] and [9] that $P(x,y)$ satisfies

$$A_xP(x,y)=\delta(x,y)+R(x,y). \tag{3}$$

Here, $R(x,y)$ is stated below,

$$R(x,y)=\frac{1}{2\pi a(y)}\sum_{i=1}^{2}\frac{x_i-y_i}{r}\frac{\partial a(x)}{\partial x_i},\, x,y\in\mathbb{R}^2, \tag{4}$$

and $\delta(x,y)$ be the Dirac delta function.

The notation r in (4) is the radius. Moreover, we define $Tu(x)$ and $T_xP(x,y)$ as given below.

$$Tu(x)=\sum_{j=1}^{2}a(x)\upsilon_j(x)\frac{\partial u(x)}{\partial x_j}, \tag{5}$$

$$T_xP(x,y)=\sum_{j=1}^{2}a(x)\upsilon_j(x)\frac{\partial P(x,y)}{\partial x_j}=\sum_{j=1}^{2}\frac{a(x)\upsilon_j(x)(x_j-y_j)}{2\pi a(y)r^2}, \tag{6}$$

where $\upsilon(x)=(\upsilon_1(x),\upsilon_2(x))$ be the outward normal of Ω.

The Dirichlet direct united BDIDEs is as read below. See, e.g., [6] and [9].

$$\begin{aligned}s(y)u(y)&+\int_{\Omega}R(x,y)u(x)\,\mathrm{d}\Omega(x)+\int_{\partial\Omega}P(x,y)Tu(x)\,\mathrm{d}\Gamma(x)\\&=\int_{\partial\Omega}\bar{u}(x)T_xP(x,y)\,\mathrm{d}\Gamma(x)+\int_{\Omega}P(x,y)f(x)\,\mathrm{d}\Omega(x),\quad y\in\bar{\Omega}.\end{aligned} \tag{7}$$

Here $\bar{\Omega}=\partial\Omega\cup\Omega$ and $c(y)$ is given by

$$s(y)=\begin{cases}1, & y\in\Omega,\\ 0, & y\in\mathbb{R}^2\setminus\bar{\Omega},\\ \alpha(y)/2\pi, & y\in\partial\Omega,\end{cases} \tag{8}$$

where $\alpha(y)$ is an interior angle at a corner point y of the boundary $\partial\Omega$.

In [6], the linear isoparametric discretization and bilinear quadrilateral mesh elements are used for the boundary $\partial\Omega$ and domain Ω, respectively. Following the same procedure, we can write

$$\partial\Omega = \bigcup_{l=1}^{L} \partial\Omega_l, \ \partial\Omega_l \cap \partial\Omega_m = \varnothing, \ l \neq m, \tag{9}$$

$$\Omega = \bigcup_{m=1}^{M} \Omega_m, \ \Omega_m \cap \Omega_k = \varnothing, \ m \neq k. \tag{10}$$

The two approaches of BDIDEs implementations are shown in [6]. The first approach is by considering the collocation points x^i in and on the boundary and interior domain such that $x^i \in \overline{\Omega}$. The second one is we consider x^i only at nodes in the interior domain i.e. $x^i \in \Omega$.

For the first approach, we take the collocation points x^i at all nodes such that $x^i \in \overline{\Omega}, \ \overline{\Omega} = \Omega \cup \partial\Omega, \ i = 1, ..., J.$ Here J is the number of nodes in and on $\partial\Omega$ such that $\partial\Omega \cup \Omega = \overline{\Omega}.$

The second approach deals with taking the collocation points for the interior nodes only i.e. $x^i \in \Omega$, $i = 1, ..., J_D$. Here J_D be the number of nodes interior to the domain Ω such that $J = J_D + J_B$, where J_B is the number of nodes on $\partial\Omega$.

The second approach gives rise to the discretized BDIDEs with reduced collocation points i.e., with $x^i \in \Omega$ as shown below. See in [6].

$$u(x^i) + \sum_{x^j \in \Omega} K_{ij}^D u(x^j) = - \sum_{x^j \in \partial\Omega} K_{ij}^D \overline{u}(x^j) + Q_i^D + D_i^D, \quad x^i \in \Omega, \tag{11}$$

where

$$K_{ij}^D = \sum_{x_j \in \overline{\Omega}_m} G_{N(j,m),i}^m + \sum_{\partial\Omega_l \subset \left\{\overline{\Omega}_m : x^j \in \overline{\Omega}_m\right\}} \tilde{A}_{N(j,m),i}^{l}, \tag{12}$$

$$Q_i^D = \sum_{l=1}^{L} \tilde{F}_i^{l}, \tag{13}$$

$$D_i^D = \sum_{m=1}^{M} H_i^m, \tag{14}$$

$$G_{Ni}^m = \int_{-1}^{1}\int_{-1}^{1} \Phi_N(\xi) R(x^i, x(\xi)) J_{m2}(\xi) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{15}$$

$$H_i^m = \int_{-1}^{1}\int_{-1}^{1} P(x^i, x(\xi)) f(x(\xi)) J_{m2}(\xi) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{16}$$

$$\tilde{A}_{N,i}^{l} = \int_{-1}^{1} P(x(\eta), \ x^i) \left[a(x(\eta)) \left(\sum_{p=1}^{2}\sum_{k=1}^{2} \frac{\partial\Phi_N(\xi)}{\partial\xi_k} \left. \frac{\partial\xi_k}{\partial x_p} \right|_{\xi=\xi(\eta)} \upsilon_p(x(\eta)) \right) \right] J_{l1}(\eta) \, \mathrm{d}\eta, \tag{17}$$

$$\tilde{F}_i^{l} = \int_{-1}^{1} \overline{u}(x(\eta)) T_x P \ (x(\eta), x^i) J_{l1}(\eta) \, \mathrm{d}\eta. \tag{18}$$

The Cartesian coordinate $x(\eta)$, $x(\eta) \in \partial\Omega_l \subset \partial\Omega$ with $-1 \le \eta \le 1$ is given as.

$$x(\eta) = \sum_{n=1}^{2} \Psi_n(\eta) X^{l\,n}, \tag{19}$$

whereas the Cartesian coordinate $x(\xi)$, $x(\xi) \in \Omega_m \subset \Omega$ with $-1 \le \xi_1 \le 1,\ -1 \le \xi_2 \le 1$ is given below.

$$x(\xi) = \sum_{N=1}^{4} \Phi_N(\xi) X^{mN}. \tag{20}$$

Here $X^{l\,n},\ n = 1,2$ and $X^{mN},\ N = 1,\ldots,4$, respectively be the nth endpoint for each line segment $\partial\Omega_l$ and the Nth vertex for each quadrilateral domain element Ω_m. The functions $\Psi_n(\eta),\ n = 1,2$ and $\Phi_N(\xi),\ N = 1,2,\ldots,4,$ are the local shape functions in respect to one-dimensional and two-dimensional, respectively, i.e.,

$$\Psi_1(\eta) = \frac{1}{2}(1-\eta), \quad \Psi_2(\eta) = \frac{1}{2}(1+\eta), -1 \le \eta \le 1. \tag{21}$$

$$\begin{aligned} &\Phi_1(\xi) = (1-\xi_1)(1-\xi_2)/4, \quad \Phi_2(\xi) = (1+\xi_1)(1-\xi_2)/4, \\ &\Phi_3(\xi) = (1+\xi_1)(1+\xi_2)/4, \quad \Phi_4(\xi) = (1-\xi_1)(1+\xi_2)/4, \quad -1 \le \xi_1, \xi_2 \le 1. \end{aligned} \tag{22}$$

The numerical implementation to obtain the BDIDEs operator for the system of equations with reduced collocation points is done by employing the Fortran compiler with double precision accuracy. After the BDIDE matrix operator is gained, we then use Matlab Software to compute the discrete eigenvalues belongs to the BDIDEs operator.

The placement of the eigenvalues for the discrete BDIDE operator is beneficial in investigating whether or not the system of equations (11) can be solved iteratively.

We denote that

$$F(x^i) = -\sum_{x^i \in \partial\Omega} K_{ij}^D \bar{u}(x^j) + Q_i^D + D_i^D, \tag{23}$$

such that (11) be written as

$$u(x^i) + \sum_{x^i \in \Omega} K_{ij}^D u(x^j) = F(x^i), \quad x^i \in \Omega. \tag{24}$$

We can also write (24) as

$$(I - \mathbf{K})u = F, \tag{25}$$

where

$$I = \delta_{ij}, \tag{26}$$

$$u = u(x^j), \tag{27}$$

$$\mathbf{K} = -K_{ij}^D, \tag{28}$$

$$F = F(x^i). \tag{29}$$

We then can write (25) as Neumann series expansion given below.

$$u = \sum_{n=0}^{N} \mathbf{K}^n F. \tag{30}$$

The calculation of $\mathbf{K}^n$ takes a lot of computation time and effort. In order to minimize those two, we denote that

$$g_0 = F, \tag{31}$$

such that

$$g_n = \mathbf{K} g_{n-1}. \tag{32}$$

Equations (31) and (32) allow us to write (30) as follows.

$$u = \sum_{n=0}^{N} \mathbf{K}^n F = F + \sum_{n=1}^{N} g_n. \tag{33}$$

We check whether the spectral radius of the operator $\mathbf{K}$ lies in a unit disc to determine the convergence of the series (30) and (33). The test domain used is a square $1 < x_1, x_2 < 2$, which is also one of the test domains used in [5]-[7]. In [6]-[7], high accuracy results for the solutions have been attained for the following interior Dirichlet problems for first and second approaches.

Observe that when $a = 1$, we will have $R(x, y) = 0$, such that Dirichlet BDIDE (7) be reduced to a BIEs, i.e.,

$$c(y)u(y) + \int_{\partial\Omega} \mathcal{F}(x,y) Tu(x)\, \mathrm{d}\Gamma(x) = \int_{\partial\Omega} \bar{u}(x) T_x \mathcal{F}(x,y)\, \mathrm{d}\Gamma(x) + \int_{\Omega} \mathcal{F}(x,y) f(x)\, \mathrm{d}\Omega(x),\ y \in \bar{\Omega}, \tag{34}$$

where $\mathcal{F}$ is the fundamental solution written by

$$\mathcal{F}(x,y) = \frac{\ln|x-y|}{2\pi}, \quad x, y \in \mathbb{R}^2. \tag{35}$$

For that, let consider a new interior Dirichlet problem, i.e., with a constant $a = 1$ i.e.
Test 1:

$$\bar{u}(x) = x_1 \text{ for } x \in \partial\Omega \text{ and } a(x) = 1,\ f(x) = 0 \text{ for } x \in \Omega \cup \partial\Omega. \tag{36}$$

We know from the theory of BIEs that all the eigenvalues of the matrix operator yields from discretized BIEs (34) with completed number of collocation points should belong to a unit circle. See e.g. [10].

Discretizing (34), letting $c(x^i) = 1$, and considering x^i only at nodes in the interior domain, i.e., $x^i \in \Omega$, for test 1, (34) is jotted down as

$$u(x^i) + \sum_{x^j \in \Omega} \kappa_{ij}^D u(x^j) = -\sum_{x^j \in \partial\Omega} \kappa_{ij}^D \bar{u}(x^j) + Q_i^D, \quad x^i \in \Omega, \tag{37}$$

where

$$\kappa_{ij}^D = \sum_{\partial\Omega_l \subset \{\bar{\Omega}_m : x^j \in \bar{\Omega}_m\}} \tilde{A}^l_{N(j,m),i}, \tag{38}$$

with Q_i^D and $\tilde{A}_{N,i}^{l}$ are given as in (13) and (17).

As previously, we can also denote that

$$F_2(x^i) = -\sum_{x^i \in \partial\Omega} \kappa_{ij}^D \bar{u}(x^j) + Q_i^D, \tag{39}$$

such that (37) be written as

$$u(x^i) + \sum_{x^i \in \Omega} \kappa_{ij}^D u(x^j) = F_2(x^i), \quad x^i \in \Omega, \tag{40}$$

which can also be written as

$$(I - \mathbf{K}_2)u = F_2 \tag{41}$$

where

$$I = \delta_{ij}, \tag{42}$$

$$u = u(x^j), \tag{43}$$

$$\mathbf{K}_2 = -\kappa_{ij}^D, \tag{44}$$

$$F_2 = F_2(x^i). \tag{45}$$

We can then write (41) as follows.

$$u = \sum_{n=0}^{N} \mathbf{K}_2^n F_2. \tag{46}$$

The calculation of $\mathbf{K}_2^n$ take a lot of computation time and effort. In order to minimize those two, we denote that

$$g_0 = F_2, \tag{47}$$

such that

$$g_n = \mathbf{K}_2^n g_{n-1}. \tag{48}$$

Equations (47) and (48) allow us to write (46) as follows.

$$u = \sum_{n=0}^{N} \mathbf{K}_2^n F_2 = F_2 + \sum_{n=1}^{N} g_n. \tag{49}$$

The Numerical Results and Discussion

In our experiment indicates that some eigenvalues for the discrete operator $\mathbf{K}_2$ with reduced number of collocation points for test 1, appears exterior to a unit circle. The explanation of why this situation occurs is presented next.

Let λ_k, for $k = 1, 2, ..., J_D$ be the eigenvalues of the matrix $\mathbf{K}_2 = -\kappa_{ij}^D$, i.e. the number λ_k that gives non-trivial solutions for the homogeneous equation $(\lambda_k I - \mathbf{K}_2)u = 0.$ We present the largest five eigen-values for the Dirichlet interior problem with $a(x)=1$ tested on a square domain $1<x_1,x_2<2$ for test 1 in the Figure 1.

The values for the number of nodes on the boundary $\partial\Omega$, J_B are taken to be $16,\ 32,\ 64$ and 128. Meshing the whole domain of the square into several sub squares yields the values of J to be $25,\ 81,\ 289$ and 1089. This implies the values of J_D are $9,\ 49,\ 225$ and 961.

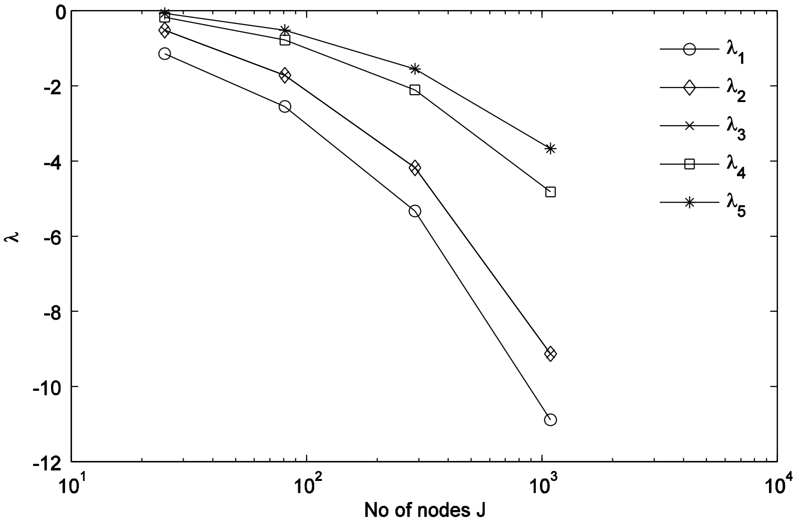

Figure 1 Eigenvalues for operator $\mathbf{K}_2$ against J

From Figure 1, as J increases, the eigenvalues of discrete operator $\mathbf{K}_2$ do not contain the unit circle. This causes the Neumann iteration in (51) to diverge. This indicates that the result obtained for BIEs with reduced number of collocation points does not agree with the theory of the BIEs that all the eigen-values of the matrix operator yields from discretized BIEs (34) with non-reduced number of collocation points should belong to a unit circle. See e.g. [10]. However, there is no theory on the eigenvalues of the Dirichlet BDIDEs with a non-reduced number of collocation points.

Observe that all the five maximal eigenvalues in Figure 1 are real numbers. There is an explanation for the results obtained from the experiment. This is due to the fact that the operator $\mathbf{K}_2$ is regarded as a discrete approximation of closed and unbounded Boundary Integral Equations (BIEs) operator. The range of operator $\mathbf{K}_2$ is the function u that maybe non-zero on $\partial\Omega$ whereas the domain of definition is the function u which is zero on $\partial\Omega$. Yet, we know from [11] that the closed unbounded operator's resolvent has an essential singularity at infinity.

Like BIEs operator with reduced number of collocation points $\mathbf{K}_2$, the BDIDEs operator with reduced number of collocation points $\mathbf{K}$ is also regarded as discrete approximation of closed and unbounded operator. Therefore, the eigenvalues distribution behaviour for BDIDEs operator $\mathbf{K}$ in (28) tested for any problem, including BVPs for PDEs with variable coefficient, is the same as the result obtained for BIEs. Therefore, the iterative method is not recommended to solve Dirichlet BDIDEs with a reduced number of collocation points.

3. CONCLUSIONS

We have presented the analysis of the spectral properties for the discrete Dirichlet BDIDEs with reduced collocation points, i.e., the collocation points x^i in the domain, $x^i \in \Omega$. The study of the behaviour of the eigenvalues for the respected operator is also discussed.

This analysis is useful in examining the behaviour and effectiveness of the iterative methods in solving discrete Dirichlet BDIDEs with reduced collocation points. Despite the fact that discrete BDIDE with $x^i \in \Omega$ requires reduced computational time for the numerical solution compared to the discrete BDIDE with $x^i \in \overline{\Omega}$, it can only be solved by direct methods.

This is from our numerical experiment that indicates the spectral radius (maximal eigenvalues) of the discrete BDIDEs operator with a reduced number of collocation points ($x^i \in \Omega$) exceeds 1. We also explained, theoretically why this condition occurs.

In conclusions, no iterative method e.g. Neumann iterative method will work for solving matrix equations of the discretized BDIDEs with reduced number of collocation points. However, any direct method in solving a linear system of equations still works in relation to the Dirichlet BDIDEs system of equations with reduced collocation points. With this finding, researchers that deal with Dirichlet BDIDEs with reduced collocation points can straightaway opt for direct methods without spending much time to solve it by using the iterations methods.

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